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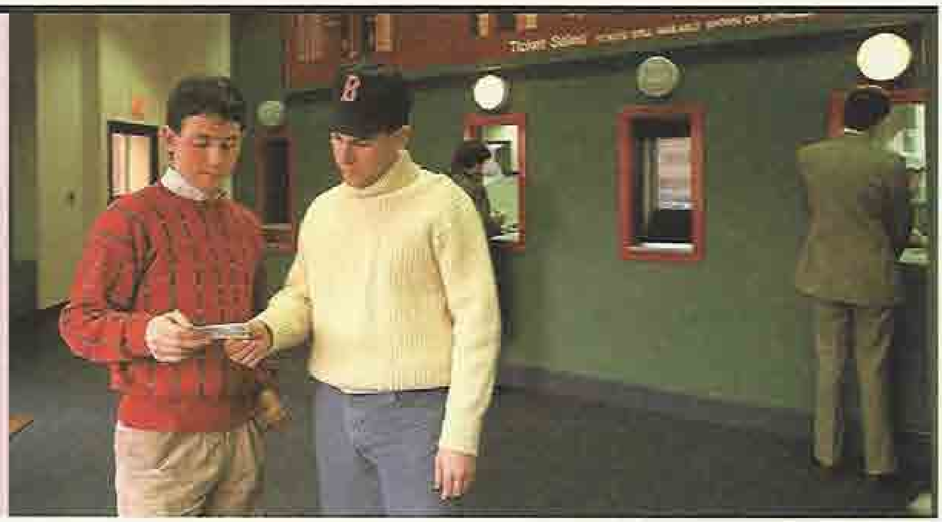
CHAPTER

8 Systems of Linear Equations

A total of 12,000 persons paid \$240,375 to attend a rock concert. If only two types of tickets were sold, one selling for \$17.50 and the other selling for \$25.00, how many of each type of tickets were sold?



8–1 ■ Systems of linear equations in two variables

In chapter 7, we learned that the graph of a linear equation in two variables is a straight line. In this chapter, we will consider the relationship that exists between two or more linear equations involving the same variables. These equations form what we call a **system of linear equations.** To illustrate, the following is a system of linear equations.

$$\begin{aligned} x + y &= 12 \\ y &= x + 6 \end{aligned}$$

A solution of a system of linear equations is any ordered pair that satisfies *both* equations. Consider the ordered pair (3,9) and the system $\begin{aligned} x + y &= 12 \\ y &= x + 6. \end{aligned}$

$$\begin{aligned} x + y &= 12 & & & y &= x + 6 \\ (3) + (9) &= 12 & &\text{Replace } x \text{ with 3 and } y \text{ with 9} & (9) &= (3) + 6 \\ 12 &= 12 & &\text{(True)} & 9 &= 9 \quad \text{(True)} \end{aligned}$$

The ordered pair (3,9) is called a *solution* of the system of linear equations. The solution set of this system is

$$\{(x,y)|x + y = 12\} \cap \{(x,y)|y = x + 6\} = \{(3,9)\}$$

Example 8–1 A

Determine if the given ordered pair is a solution of the system of linear equations.

1. $(x,y) = (2,5)$ and the system $x + 2y = 12$
$3x - y = 1$.

$$\begin{aligned} x + 2y &= 12 \\ (2) + 2(5) &= 12 \\ 2 + 10 &= 12 \\ 12 &= 12 \quad \text{(True)} \end{aligned} \qquad \begin{aligned} 3x - y &= 1 \\ 3(2) - (5) &= 1 \quad \text{Replace } x \text{ with 2 and } y \text{ with 5} \\ 6 - 5 &= 1 \\ 1 &= 1 \quad \text{(True)} \end{aligned}$$

The ordered pair (2,5) is a solution of the system of linear equations since it satisfies both of the original equations.

2. $(x,y) = (-5,-2)$ and the system $3x - 2y = -11$
$2x + y = 10$.

$$\begin{aligned} 3x - 2y &= -11 \\ 3(-5) - 2(-2) &= -11 \\ -15 + 4 &= -11 \\ -11 &= -11 \quad \text{(True)} \end{aligned} \qquad \begin{aligned} 2x + y &= 10 \\ 2(-5) + (-2) &= 10 \quad \text{Replace } x \text{ with } -5 \text{ and } y \text{ with } -2 \\ -10 + (-2) &= 10 \\ -12 &= 10 \quad \text{(False)} \end{aligned}$$

The ordered pair $(-5,-2)$ is *not* a solution of the system of linear equations since it does not satisfy both equations.

▶ ***Quick check*** Determine if $(3,-1)$ is a solution of the system
$2x - y = 7$
$x - 5y = 2$.

If we wish to represent a system of linear equations graphically, the graph will be a pair of straight lines. Two straight lines, L_1 and L_2, can be related in one of three ways. See figure 8–1.

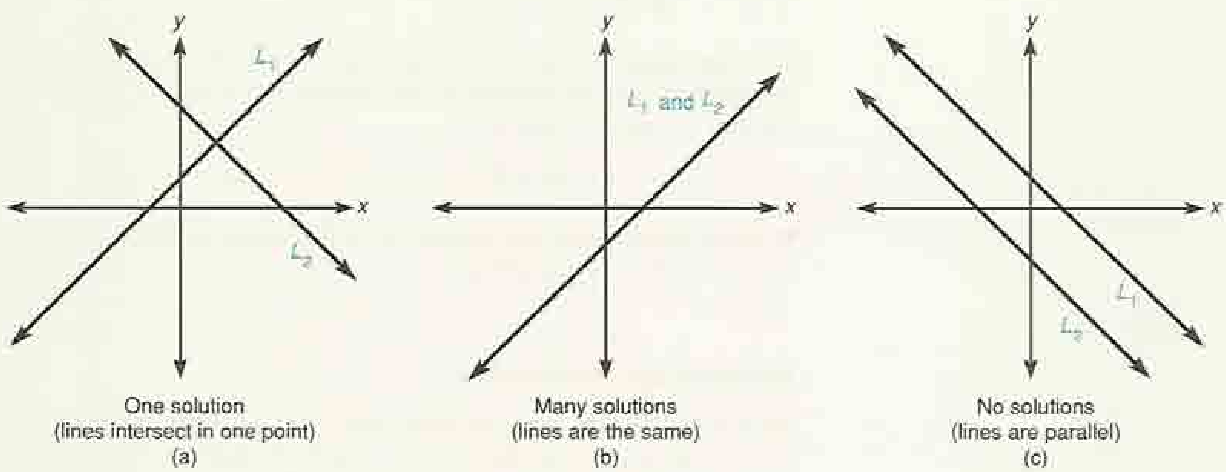


Figure 8–1

Graphs of systems of linear equations

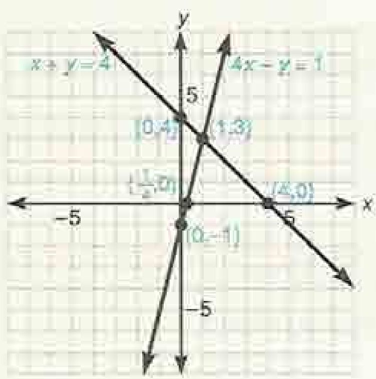
1. The graphs intersect in a single point [figure 8–1(a)]. The solution is the point of intersection. This system is called **consistent and independent.**
2. The graph is the same line [figure 8–1(b)]. Any solution of one equation is a solution of the other equation. The system is called **consistent and dependent.**
3. The graphs are parallel lines [figure 8–1(c)]. There is no solution and the system is called **inconsistent.**

Graphing systems of linear equations can be used only as a good estimate of the solution.

Example 8–1 B

Solve the system of equations $4x - y = 1$ by graphing.
$$x + y = 4$$

Using the x- and y-intercepts, we graph each equation.



The lines appear to intersect at the point (1,3), so the system is consistent and independent. The intersection of the solution sets is the set $\{(1,3)\}$ which we can write in set-builder notation.

$$\{(x,y)|4x - y = 1\} \cap \{(x,y)|x + y = 4\}$$

▶ ***Quick check*** Find the solution set of the system of linear equations by graphing.
$4x - y = 0$
$x + y = 5$

Solution by elimination

Since it is time-consuming to graph and it can be difficult to read exact solutions from the graph, we usually use other, algebraic, methods to solve a system. One such method is called the **elimination** (or **addition**) method. The following examples demonstrate this method in which we eliminate one of the variables through addition. We use the property that when the same expression is added to both members of an equation, the results are equal.

■ Example 8–1 C

Find the solution set of the following systems by elimination.

1. $4x - 3y = 5 \quad (1)$
   $2x + 3y = 7 \quad (2)$

   The elimination method involves adding the respective members of the two equations so that one of the variables is eliminated.

$$\begin{array}{rll} 4x - 3y &= 5 & \\ \underline{2x + 3y} &\underline{= 7} & \\ 6x \qquad &= 12 & \text{Add left and right members} \\ x &= 2 & \text{Divide each member by 6} \end{array}$$

   We now replace x with 2 in either equation and solve for y.

   Using $4x - 3y = 5$,

$$\begin{array}{rll} 4(2) - 3y &= 5 & \text{Replace } x \text{ with 2} \\ 8 - 3y &= 5 & \text{Multiply as indicated} \\ -3y &= -3 & \text{Add } -8 \text{ to each member} \\ y &= 1 & \text{Divide each member by } -3 \end{array}$$

   The simultaneous solution of the system is $x = 2$ and $y = 1$, which we write as the ordered pair $(2,1)$. The solution set is $\{(2,1)\}$, which can be checked by substituting into each equation.

$$\begin{array}{lrl lrll} (1) & 4x - 3y &= 5 & (2) & 2x + 3y &= 7 & \\ & 4(2) - 3(1) &= 5 & & 2(2) + 3(1) &= 7 & \text{Replace } x \text{ with 2 and } y \text{ with 1} \\ & 8 - 3 &= 5 & & 4 + 3 &= 7 & \\ & 5 &= 5 \ \text{(True)} & & 7 &= 7 & \text{(True)} \end{array}$$

Note We will not check our solution in future examples, but this is something we should get into the habit of doing automatically.

We *eliminated* one variable (in this case y) through *addition*. This was accomplished because the *coefficients* of y are *additive inverses* (in this case 3 and -3). We then obtain a linear equation in one variable that we can easily solve. Sometimes it is necessary to multiply the terms of one equation, or both equations, by some real number to get additive inverse coefficients of one of the variables before adding to eliminate that variable.

2. $3x = 5y - 1 \quad (1)$
   $2x - 3y = 8 \quad (2)$

   We must first write equation (1) in standard form.

   $3x - 5y = -1$ Add $-5y$ to each member

   The system now states

   $3x - 5y = -1$ New equation (1)
   $2x - 3y = 8$

   Adding the corresponding members of these equations yields the equation $5x - 8y = 7$ and no variable has been eliminated. We must multiply by the appropriate number(s) to eliminate either x or y when the equations are added.

One way to eliminate one variable in this case (and there are other ways) is to multiply new equation (1) by 2 and equation (2) by -3.

$$\begin{array}{lll} 3x - 5y = -1 & \text{Multiply by 2} \rightarrow & 6x - 10y = -2 \\ 2x - 3y = 8 & \text{Multiply by } -3 \rightarrow & \underline{-6x + 9y = -24} \\ & \text{Add} & -y = -26 \\ & & y = 26 \end{array} \left.\right\} \text{Coefficients of } x \text{ are additive inverses}$$

Substitute 26 for y in equation (2) and solve for x.

$$\begin{aligned} 2x - 3(26) &= 8 && \text{Replace } y \text{ with 26} \\ 2x - 78 &= 8 && \text{Solve for } x \\ 2x &= 86 \\ x &= 43 \end{aligned}$$

The solution set is $\{(43,26)\}$.

Note We could have multiplied new equation (1) by 3 and equation (2) by -5, or equation (1) by -3 and equation (2) by 5. In either case, we would eliminate y through adding.

Occasionally, when we solve a system we find the system to be *inconsistent* (no solutions) or *dependent* (many solutions).

3. $3x - 4y = 1 \qquad (1)$
$6x - 8y = -5 \qquad (2)$

$$\begin{array}{lll} 3x - 4y = 1 & \text{Multiply by } -2 \rightarrow & -6x + 8y = -2 \\ 6x - 8y = -5 & & \underline{6x - 8y = -5} \\ & & 0 = -7 \quad \text{Add members} \end{array} \left.\right\} \text{Coefficients of } x \text{ and } y \text{ are additive inverses}$$

We obtain a *false* statement. This indicates the system has *no solution*. The system is *inconsistent* and the solution set is $\emptyset$. The lines are parallel.

4. $3x - 2y = 1 \qquad (1)$
$9x - 6y = 3 \qquad (2)$

$$\begin{array}{lll} 3x - 2y = 1 & \text{Multiply by } -3 \rightarrow & -9x + 6y = -3 \\ 9x - 6y = 3 & & \underline{9x - 6y = 3} \\ & & 0 = 0 \quad \text{Add members} \end{array} \left.\right\} \text{Coefficients of } x \text{ and } y \text{ are additive inverses}$$

The resulting statement is *true*, which indicates that every solution of one equation is also a solution of the other equation. The system is *dependent* and the solution set is

$\{(x,y)|3x - 2y = 1\}$ or $\{(x,y)|9x - 6y = 3\}$

The lines are the same.

▶ ***Quick check*** Find the solution set of the linear system $\begin{aligned} 2x - 3y &= 2 \\ 3x + 2y &= 3 \end{aligned}$ by elimination.

To summarize, the elimination method of solving a system of two linear equations in two variables, we use the following procedure.

To solve linear systems of equations by elimination

1. Write the system so that each equation is in standard form $ax + by = c$.
2. Multiply one equation (or both equations), if necessary, by a number to obtain additive inverse coefficients of one of the variables.
3. Add the corresponding members of the resulting equations and solve the resulting equation in *one variable*.
4. Substitute the value of the variable obtained into one of the *original* equations and solve for the other variable.
5. In step 3, if we get
   a. a false statement, the system is inconsistent, there are no solutions, and the lines are parallel. The solution set is $\emptyset$.
   b. the equation $0 = 0$, the system is consistent and dependent, there are infinitely many solutions, and the lines are the same.

Solution by substitution

A second algebraic method that is used to solve a system of linear equations involves solving one of the equations for one of the variables and substituting for that variable into the other equation to obtain an equation in one variable. We call this the **substitution** method for solving a system of linear equations.

■ *Example 8–1 D*

Find the solution set of the following systems by substitution.

1. $4x + 3y = 14$ (1)

$y = 3x - 4$ (2)

Notice that equation (2) is solved for y. Substitute $3x - 4$ for y in equation (1).

$$
\begin{aligned}
4x + 3(3x - 4) &= 14 && \text{Replace } y \text{ with } 3x - 4\\
4x + 9x - 12 &= 14 && \text{Multiply in left member}\\
13x - 12 &= 14\\
13x &= 26\\
x &= 2
\end{aligned}
$$

Since $y = 3x - 4$, substitute 2 for x in this equation.

$$
\begin{aligned}
y &= 3(2) - 4 && \text{Replace } x \text{ with } 2\\
y &= 6 - 4\\
y &= 2
\end{aligned}
$$

The solution set is $\{(2,2)\}$.

Check the solution by substituting 2 for x and 2 for y in both equations.

2. $x - 3y = 4$ (1)
$3x + 4y = 1$ (2)

Solving the equation $x - 3y = 4$ for x, we have the system

$$\begin{aligned} x &= 3y + 4 && \text{Add } 3y \text{ to each member of (1)} \\ 3x + 4y &= 1 \end{aligned}$$

Substituting $3y + 4$ for x into the equation $3x + 4y = 1$, we obtain

$$\begin{aligned} 3(3y + 4) + 4y &= 1 && \text{Replace } x \text{ with } 3y + 4 \\ 9y + 12 + 4y &= 1 \\ 13y + 12 &= 1 \\ 13y &= -11 \\ y &= -\frac{11}{13} \end{aligned}$$

To find x, substitute $-\frac{11}{13}$ for y into one of the original equations, say $x - 3y = 4$, then

$$\begin{aligned} x - 3\left(-\frac{11}{13}\right) &= 4 && \text{Replace } y \text{ with } -\frac{11}{13} \\ x + \frac{33}{13} &= 4 \\ x &= 4 - \frac{33}{13} = \frac{52}{13} - \frac{33}{13} \\ x &= \frac{19}{13} \end{aligned}$$

The solution set is $\left\{\left(\frac{19}{13}, -\frac{11}{13}\right)\right\}$.

▶ ***Quick check*** Find the solution set of the system $\begin{aligned} y - 3x &= 2 \\ 2x + 5y &= -7 \end{aligned}$ by substitution.

To summarize, the substitution method of solving a system of two linear equations in two variables, we use the following procedure.

To solve systems of linear equations by substitution

1. Solve one of the equations for one of the variables. (If one of the variables has a coefficient of 1 or -1, solve for it.)
2. Substitute the expression obtained in step 1 for that variable in the other equation, and solve the resulting equation in one variable.
3. Substitute the value for the variable into the equation obtained in step 1 and solve for the other variable.
4. In step 2,
   a. if we get a false statement, the solution set is $\emptyset$, the system is inconsistent, and the lines are parallel.
   b. if we get a statement that is *always* true, the solution set is the solution set of either equation, the system is dependent, and the lines are the same.

Mastery points

Can you

- Determine whether an ordered pair is a solution of a system?
- Solve a system of linear equations in two variables by graphing?
- Solve a system of linear equations in two variables using elimination?
- Solve a system of linear equations in two variables using substitution?

Exercise 8–1

Determine whether the given ordered pair is a simultaneous solution of the system of linear equations. See example 8–1 A.

Example $2x - y = 7$
$x - 5y = 2$; $(3,-1)$

Solution

$$\begin{aligned} 2x - y &= 7 \\ 2(3) - (-1) &= 7 \\ 6 + 1 &= 7 \\ 7 &= 7 \quad \text{(True)} \end{aligned} \qquad \begin{aligned} x - 5y &= 2 \\ (3) - 5(-1) &= 2 \\ 3 + 5 &= 2 \\ 8 &= 2 \quad \text{(False)} \end{aligned}$$

Original equation
Replace x with 3 and y with -1

The ordered pair $(3,-1)$ is not a solution of the system since it does not satisfy *both* equations.

1. $x + y = 6$
   $x - y = -2$; $(2,4)$
2. $3x - y = 1$
   $x + y = -5$; $(-1,-4)$
3. $2x + 3y = 6$
   $x - 2y = 3$; $(3,0)$
4. $x - 6y = 0$
   $2x + 3y = 8$; $(4,1)$
5. $3x - 5y = -12$
   $x - y = -7$; $(-1,2)$
6. $5x + 8y = 32$
   $9x - 2y = -8$; $(0,4)$

Find the solution set of the following systems of linear equations by graphing. If the system is inconsistent or dependent, so state. See example 8–1 B.

Example $4x - y = 0$ (1)
$x + y = 5$ (2)

Solution We graph each equation on the same rectangular coordinate plane using the slope and intercept for equation (1) and the x- and y-intercepts for equation (2)

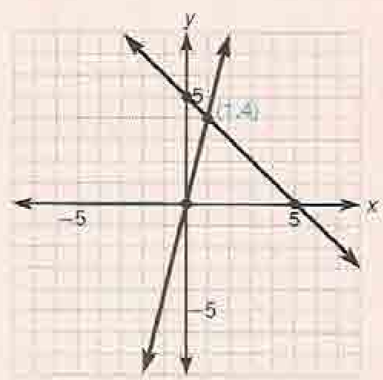
(1) $y = 4x + 0$

$m = 4 = \dfrac{4}{1}$ and $b = 0$

(2) $x + y = 5$

x-intercept, $(5,0)$

y-intercept, $(0,5)$



The graphs intersect at the point $(1,4)$ so the solution set is $\{(1,4)\}$.

$2x - y = 4$
$x + y = 5$

8. $2x + 3y = 12$
$x - y = 1$

9. $2x - y = -5$
$x + 2y = 0$

10. $6x - 2y = -8$
$3x - y = 4$

11. $x - 2y = 6$
$-2x + 4y = -12$

12. $2x - y = -1$
$3x - y = -1$

Find the solution set of each system of linear equations by elimination. If the equations are inconsistent or dependent, so state. See example 8–1 C.

Example $2x - 3y = 2$
$3x + 2y = 3$

Solution

$$\begin{array}{lll} 2x - 3y = 2 & \text{Multiply by } 2 \rightarrow & 4x - 6y = 4 \quad \text{Coefficients of } y \text{ are opposites} \\ 3x + 2y = 3 & \text{Multiply by } 3 \rightarrow & \underline{9x + 6y = 9} \\ & & 13x \quad = 13 \quad \text{Add members} \\ & & x = 1 \quad \text{Divide by 13} \end{array}$$

Using $2x - 3y = 2$,

$$\begin{aligned} 2(1) - 3y &= 2 && \text{Replace } x \text{ with } 1 \\ 2 - 3y &= 2 && \text{Solve for } y \\ -3y &= 0 \\ y &= 0 \end{aligned}$$

The solution set is $\{(1,0)\}$.

13. $x - y = 3$
$x + y = -7$

14. $3x + y = 1$
$2x - y = 9$

15. $x + 4y = 10$
$x + 2y = 4$

16. $x - y = 3$
$3x - y = -7$

17. $5x - 3y = 1$
$2x - 3y = -5$

18. $-5x - y = 4$
$-5x + 2y = 7$

19. $3x + 2y = 11$
$x - y = 2$

20. $3x + 4y = 18$
$2x - y = 1$

21. $4x + y = 7$
$2x + 3y = 6$

22. $4x - 3y = 7$
$3x - 2y = 6$

23. $8x - y = -4$
$4x + 7y = -32$

24. $3x - y = 10$
$6x - 2y = 5$

25. $3x + y = 2$
$9x + 3y = 6$

26. $-x + 2y = -7$
$-2x + 6y = 1$

27. $6x - 5y = 0$
$12x - 10y = -3$

28. $x - y = 9$
$-4x + 4y = -36$

29. $4x - 2y = 0$
$3x + 3y = 5$

30. $-2x + y = 6$
$4x + y = 1$

31. $\frac{1}{2}x + \frac{1}{3}y = 1$
$\frac{1}{4}x - \frac{2}{3}y = \frac{1}{12}$

32. $\frac{2}{3}x - \frac{1}{4}y = 4$
$x + \frac{1}{3}y = 2$

33. $\frac{3}{2}x + \frac{2}{5}y = \frac{9}{10}$
$\frac{1}{2}x + \frac{6}{5}y = \frac{3}{10}$

34. $\frac{5}{7}x - \frac{4}{5}y = \frac{9}{10}$
$\frac{2}{7}x - \frac{2}{5}y = \frac{3}{10}$

35. $\frac{5}{2}x - y = \frac{-17}{2}$
$\frac{2}{3}x + \frac{2}{3}y = 1$

36. $(0.3)x - (0.8)y = 1.6$
$(0.1)x + (0.4)y = 1.2$

37. $(0.1)x + (0.3)y = -0.3$
$(0.5)x + (0.4)y = 0.7$

38. $x - (0.5)y = 3$
$(0.9)x + y = -0.2$

Find the solution set of the following systems of linear equations by substitution. If the system is dependent or inconsistent, so state. See example 8–1 D.

Example $y - 3x = 2$
$2x + 5y = -7$

Solution Solve the equation $y - 3x = 2$ for y.

$$\begin{aligned} y - 3x &= 2 \\ y &= 3x + 2 \quad \text{Add } 3x \text{ to each member} \end{aligned}$$

Substitute $3x + 2$ for y in $2x + 5y = -7$ and solve for x.

$$\begin{aligned} 2x + 5(3x + 2) &= -7 \quad \text{Replace } y \text{ with } 3x + 2 \\ 2x + 15x + 10 &= -7 \quad \text{Solve the resulting equation} \\ 17x + 10 &= -7 \\ 17x &= -17 \\ x &= -1 \end{aligned}$$

Substitute -1 for x in $y - 3x = 2$.

$$\begin{aligned} y - 3x &= 2 \\ y - 3(-1) &= 2 \quad \text{Replace } x \text{ with } -1 \\ y + 3 &= 2 \\ y &= -1 \end{aligned}$$

The solution set is $\{(-1,-1)\}$.

39. $2x + y = 10$, $y = -x + 3$
40. $3x + 2y = 9$, $y = 2x - 3$
41. $x = y + 5$, $x + 5y = -4$
42. $5x + y = 10$, $x = -2 + 3y$
43. $y = 3 - 6x$, $2x + 5y = -13$
44. $3x - 5y = 4$, $x + 2y = -2$
45. $x - y = 3$, $2x + 2y = -10$
46. $3x + 3y = 0$, $x + y = 5$
47. $5x - y = 8$, $2x - y = -4$
48. $-x + 3y = 4$, $-x - 8y = 1$
49. $4x - 3y = 7$, $y = -5$
50. $2x + 7y = 8$, $y = 1$
51. $5x - 6y = -6$, $x = 6$
52. $7x + 6y = -9$, $x = -8$
53. $-4x - 3y = 4$, $y = 0$

Solve each system of linear equations by either elimination or substitution. Try to choose the most suitable method. If the system is dependent or inconsistent, so state. See examples 8–1 C and D.

54. $-6x + 3y = 4$, $-12x - 6y = 1$
55. $3x + 2y = -6$, $-6x - 4y = 1$
56. $5x - 10y = 5$, $x - 2y = 1$
57. $2x - y = 7$, $6x - 3y = 21$
58. $2x - 3y = 5$, $3x + 4y = 1$
59. $4x - 3y = 4$, $-2x + 4y = 3$
60. $7x - 8y = 14$, $5x + 3y = -4$
61. $-\frac{1}{3}x + y = 4$, $x = \frac{1}{3}y - 1$
62. $\frac{2}{3}x - \frac{1}{3}y = 4$, $x = \frac{1}{3}y - 1$
63. $\frac{2}{5}x - \frac{3}{5}y = 3$, $y = \frac{5}{2}x - 3$
64. $\frac{1}{6}x + y = -7$, $y = \frac{-2}{3}x + 2$

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Find the solution set of the following systems. Let $p = \frac{1}{x}$ and $q = \frac{1}{y}$. Substitute, solve for p and q, then solve for x and y.

65. $\frac{1}{x} + \frac{1}{y} = 3$
$\frac{1}{x} - \frac{1}{y} = -5$

$\boxed{66.}$ $\frac{2}{x} - \frac{3}{y} = 1$
$\frac{1}{x} + \frac{2}{y} = 2$

67. $\frac{4}{x} + \frac{5}{y} = 0$
$\frac{3}{x} + \frac{2}{y} = 1$

68. $\frac{-3}{x} + \frac{1}{y} = 5$
$\frac{2}{x} - \frac{4}{y} = -1$

Using the variables x and y, write an algebraic equation for each verbal statement.

Example The sum of two numbers is 36.

Solution Let x = one of the numbers and y = the other number.
The algebraic equation is $x + y = 36$.

69. The sum of two numbers is 502.

70. Jane invested a total of \$4,000 in two accounts.

71. One number is 6 more than another number.

72. One number is 3 less than twice a second number.

73. The length of a rectangle is 4 more than three times the width.

74. The length of a rectangle is 5 less than twice the width.

75. The difference in two electric currents is 33 amperes.

76. A boat takes $3\frac{1}{2}$ hours to go from point A to point B and back.

Review exercises

In problems 1–3, perform the indicated operations. See section 3–2.

1. $(4x^3 - 2x^2 + 1) - (5x^3 + 3x^2 + x - 4)$
2. $(x + 2)(x^2 - 2x + 4)$
3. $(5y - 2z)^2$
4. Find the solution set of the radical equation $\sqrt{x}\sqrt{x + 8} = 3$. See section 6–5.
5. One number is 27 more than another number. The smaller number is one-fourth of the larger number. Find the numbers. See section 2–3.
6. If three times a number is increased by 15, the result is 51. What is the number? See section 2–3.

8–2 ■ Applied problems using systems of linear equations

Being able to solve a system of linear equations lets us solve many word problems using two variables as opposed to the one-variable approach we used in chapter 2.

■ Example 8–2 A

1. The perimeter of a rectangular plot of ground is 238 meters. If the length of the rectangle is 11 meters longer than twice the width, what are the dimensions of the rectangle? [Use perimeter = 2(length) + 2(width).]

Let w = the width of the rectangle and ℓ = the length of the rectangle.

length of the rectangle	is	11 meters	longer than	twice the width
ℓ	=	11	+	$2 \cdot w$

Thus $\ell = 2w + 11$.

We get the second equation by using the formula $P = 2\ell + 2w$ and the given information that $P = 238$.

$(238) = 2\ell + 2w$ Replace P with 238

We form the system of linear equations $\ell = 2w + 11$
$2\ell + 2w = 238$

Since one equation is already solved for ℓ, we use the substitution method.

$$\begin{aligned} 2(2w + 11) + 2w &= 238 && \text{Replace } \ell \text{ with } 2w + 11 \text{ in } 2\ell + 2w = 238 \\ 4w + 22 + 2w &= 238 && \text{Solve for } w \\ 6w + 22 &= 238 \\ 6w &= 216 \\ w &= 36 \end{aligned}$$

$$\begin{aligned} \text{Now } \ell &= 2w + 11 \\ &= 2(36) + 11 && \text{Replace } w \text{ with } 36 \\ &= 72 + 11 \\ &= 83 \end{aligned}$$

The rectangle is 36 meters wide and 83 meters long.

Check: 1. $2(36) + 2(83) = 72 + 166 = 238$ (True)
2. (83) is 11 more than $2(36)$ (True)

Note We will not show checks in the future. However, you should always check your answers.

2. Arlene wishes to invest \$5,000. If she invests part at 7% simple interest, part at 6% simple interest, and receives a total interest of \$332 after one year, how much does she invest at each rate?

Note We use the formula $i = prt$ where i is the simple interest received when principal p is invested at rate r for t years. Time will always be one year in these problems, so we may use the simplified formula $i = pr$.

Let x = the amount invested at 7% and y = the amount invested at 6%. We use the following data table to set up equations.

	Amount invested	Rate of return	Amount of return
First investment	x	7%	$0.07x$
Second investment	y	6%	$0.06y$
Total investment	5,000		332

The equations are found by reading *down* the data table (see arrows). The system of linear equations is then

$$\begin{aligned} x + y &= 5{,}000 \\ 0.07x + 0.06y &= 332 \end{aligned}$$

To eliminate decimal numbers in the second equation, multiply each term by 100. Then, we have

$$\begin{aligned} x + y &= 5{,}000 \quad \text{Multiply by } -7 \rightarrow & -7x - 7y &= -35{,}000 \\ 7x + 6y &= 33{,}200 & 7x + 6y &= 33{,}200 \\ & & -y &= -1{,}800 \quad \text{Add members} \\ & & y &= 1{,}800 \quad \text{Multiply by } -1 \end{aligned}$$

Since $x + y = 5{,}000$, then

$$\begin{aligned} x + (1{,}800) &= 5{,}000 \quad \text{Replace } y \text{ with } 1{,}800 \\ x &= 3{,}200 \end{aligned}$$

Arlene invests \$1,800 at 6% simple interest and \$3,200 at 7% simple interest.

▶ ***Quick check*** a. The perimeter of a rectangular flower garden is 82 feet. If the length of the rectangle is 5 feet longer than twice the width, what are the dimensions of the rectangle?
b. Jim wishes to invest \$7,500. If he invests part at 6% interest, part at 8% interest, and receives \$536 total interest after one year, how much did he invest at each rate?

How to solve a word problem using a system of linear equations

1. Read the problem carefully and completely. Note what information is given and what information we wish to find.
2. Whenever possible, draw a diagram showing the relationships in the problem.
3. Choose different variables for each unknown quantity.
4. Use word statements in the problem to write a system of linear equations. Generally, there are *as many equations as there are different variables.*
5. Solve the system of linear equations using one of the methods you have learned.
6. Check the results in the original statement of the problem.

Mastery points

Can you

- Solve a word problem using systems of linear equations in two variables?

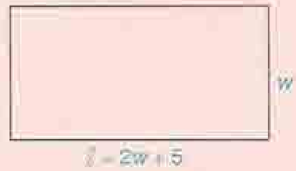
Exercise 8–2

Set up a system of two linear equations and solve. See example 8–2 A–1.

Example The perimeter of a rectangular flower garden is 82 feet. If the length of the rectangle is 5 feet longer than twice the width, what are the dimensions of the rectangle?

Solution Let w = the width of the rectangle and ℓ = the length of the rectangle.

$$\begin{array}{ccccc} \text{length} & \text{is} & \text{5 feet} & \text{longer than} & \text{twice the width} \\ \ell & = & 5 & + & 2 \cdot w \end{array}$$



Using $P = 2\ell + 2w$, given $P = 82$, then

$2\ell + 2w = (82)$ Replace P with 82
$\ell = 2w + 5$

Solving the system using substitution, substitute $2w + 5$ for ℓ in the equation $2\ell + 2w = 82$.

$$\begin{aligned} 2(2w + 5) + 2w &= 82 && \text{Replace } \ell \text{ with } 2w + 5 \\ 4w + 10 + 2w &= 82 && \text{Multiply in left member} \\ 6w + 10 &= 82 && \text{Combine like terms} \\ 6w &= 72 && \text{Add } -10 \text{ to each member} \\ w &= 12 && \text{Divide each member by 6} \end{aligned}$$

$$\begin{aligned} \ell &= 2w + 5 \\ \ell &= 2(12) + 5 && \text{Replace } w \text{ with 12 and solve for } \ell \\ \ell &= 24 + 5 \\ \ell &= 29 \end{aligned}$$

The dimensions of the rectangle are 12 feet wide and 29 feet long.

1. If twice the length of a rectangular floor is increased by three times the width, the sum is 48 feet. The perimeter of the room is 40 feet. What are the dimensions of the floor?
2. The distance around a rectangular flower garden is 64 feet. If the length is three times the width, what are the dimensions?
3. The perimeter of a rectangular plot of ground is 30 meters. Three times the length minus four times the width is 3 meters. Find the length and the width of the plot.
4. A 20-foot board must be cut into two pieces so that one piece is 4 feet longer than the other piece. How long is each piece?
5. A 21-foot piece of pipe must be cut into two pieces so that one piece is 9 feet longer than the other piece. How long is each piece of pipe?
6. The sum of two electric currents is 96 amps. If one current is 22 amps less than the other, how many amps are there in each current?
7. The sum of two voltages in an electric circuit is 47 volts and their difference is 25 volts. Find the voltages.
8. The difference in the number of teeth in two gears is 14 and their sum is 72. How many teeth are there in each gear?

See example 8–2 A–2.

Example Jim wishes to invest \$7,500. If he invests part at 6% interest, part at 8% interest, and receives \$536 total interest after one year, how much did he invest at each rate?

Solution Let x = the amount invested at 6% interest and y = the amount invested at 8% interest.

	Amount invested	Rate of return	Amount of return
First investment	x	6%	$0.06x$
Second investment	y	8%	$0.08y$
Total investment	7,500		536

The system of linear equations is

$$\begin{aligned} x + y &= 7{,}500 \\ 0.06x + 0.08y &= 536 \end{aligned}$$

Multiply the terms of the second equation by 100.

$$\begin{aligned} x + y &= 7{,}500 \\ 6x + 8y &= 53{,}600 \end{aligned} \quad \xrightarrow{\text{Multiply by } -8} \quad \begin{aligned} -8x - 8y &= -60{,}000 \\ 6x + 8y &= 53{,}600 \\ \hline -2x &= -6{,}400 \quad \text{Add members} \\ x &= 3{,}200 \end{aligned}$$

Using $x + y = 7{,}500$,

$$\begin{aligned} (3{,}200) + y &= 7{,}500 \quad \text{Replace } x \text{ with } 3{,}200 \\ y &= 4{,}300 \end{aligned}$$

Jim invested \$3,200 at 6% interest and \$4,300 at 8% interest.

9. Phoebe wishes to invest \$20,000, part at 7% interest and the rest at $6\frac{1}{2}\%$. If her total income from the two investments for one year is \$1,370, how much should she invest at each rate?
10. The income from two investments for one year is \$1,485. If \$19,000 is invested, part at 8% and the rest at $7\frac{1}{2}\%$, how much is invested at each rate?
11. The income from an 8% investment is \$300 more than the income from a 6% investment. How much is invested at each rate if a total of \$30,000 is invested?
12. Jamie invests a total of \$16,000, part at 7% interest and part at 9% interest. If the income from each investment is the same, how much money does Jamie invest at each rate?
13. The income from two investments is the same. If \$36,000 is invested, part at 7% and the rest at 8%, how much is invested at each rate?
14. Juanita invests a total of \$22,000. She suffers a net yearly loss of \$160 on her two investments. If one investment made her a 12% profit and the other investment caused an 8% loss, how much was in each investment?
15. In one year, Simone makes a 15% profit on one investment but takes a 12% loss on a second investment. If she invests a total of \$25,000 and realizes a net gain of \$1,320 on her two investments, how much is each investment?

Example Sam's Pizza Emporium sells two kinds of large pizzas at \$6.50 and \$8.00 per pizza. If Sam sells 43 large pizzas for \$306.50, how many of each kind of pizza does he sell?

Solution Let x = number of \$6.50 pizzas sold and y = number of \$8.00 pizzas sold.
We now set up a table to help us determine the equations.

	Number of pizzas	Cost per pizza	Amount of income
\$6.50 pizzas	x	6.50	$6.50x$
\$8.00 pizzas	y	8.00	$8.00y$
Total number	43		306.50

The system of linear equations is then

$$\begin{aligned} x + y &= 43 \\ 6.50x + 8.00y &= 306.50 \end{aligned}$$

Multiply the second equation by 10 to clear decimal numbers.

$$\begin{aligned} x + y &= 43 \quad \xrightarrow{\text{Multiply by } -65} & -65x - 65y &= -2{,}795 \\ 65x + 80y &= 3{,}065 & 65x + 80y &= 3{,}065 \\ & & 15y &= 270 \quad \text{Add members} \\ & & y &= 18 \end{aligned}$$

Using the equation $x + y = 43$,

$$\begin{aligned} x + y &= 43 \\ x + (18) &= 43 \quad \text{Replace } y \text{ with } 18 \\ x &= 25 \end{aligned}$$

Sam sold 25 pizzas at \$6.50 and 18 pizzas at \$8.00.

16. A keypunch operator at a local firm works for \$9 per hour and an entry-level typist works for \$6.50 per hour. The total pay for an 8-hour day is \$476, and there are two more typists than keypunch operators. How many keypunch operators does the firm employ?

17. A clothing store sells men's suits at \$152 and \$205 per suit. The store sells 32 suits and takes in \$5,659. How many suits at each price does the store sell?

18. A hardware supply company sells two types of doorknobs. The chromium-plated knob sells at \$8 per knob, and the solid brass knob sells for \$11.50 per knob. The company sold 420 doorknobs for \$3,622.50. How many of each type were sold?

19. A road construction crew consists of cat operators working at \$90 per day and laborers working at \$50 per day. The total payroll per day is \$1,600. If there are 3 laborers doing odd jobs and 4 laborers are assigned to work with each cat operator, how many laborers are there in the crew?

20. Skilled and unskilled workers are employed by a construction firm. If 5 skilled workers and 8 unskilled workers are employed, the total wages per day are \$948. When 3 skilled and 5 unskilled workers are employed, the total wages per day are \$580. What is the daily rate of pay of each type of worker?

21. The tickets for a puppet show cost \$3.50 for adults and \$1.25 for children. If \$853.75 in tickets are sold to an audience of 503, how many children's tickets were sold?

22. A movie theater sold 323 tickets for \$831.50. If adult tickets cost \$3 and children's tickets cost \$1.75, how many tickets of each type were sold?

23. Fernando has saved 43 coins in dimes and quarters. If he has saved a total of \$7.15, how many dimes and how many quarters does he have?

24. Pam has a collection of fifty-two 13-cent and 20-cent stamps. If the face value of her collection is \$9.28, how many 20-cent stamps does she have?

Example An auto mechanic has two bottles of battery acid. One bottle contains a 10% acid solution and the other contains a 4% acid solution. How many cubic centimeters (cm^3) of each solution are needed to make 120 cubic centimeters of a 6% acid solution?

Solution Let x = the number of cubic centimeters of 10% acid solution and
y = the number of cubic centimeters of 4% acid solution.

Note 10% acid solution means that 10% of the solution is acid and the rest, 90%, is water. Thus, the 10% acid solution has

$$(0.10)(x) = 0.10x \text{ cu. cm of acid}$$
$$(0.90)(x) = 0.90x \text{ cu. cm of water}$$

	Volume	% acid	Amount of acid
First solution	x	10	$0.10x$
Second solution	y	4	$0.04y$
Total mixture	120	6	$0.06(120) = 72$

The system of linear equations is

$$\begin{aligned} x + y &= 120 \\ 0.10x + 0.04y &= 72 \end{aligned}$$

To clear the decimal numbers in the second equation, multiply each term by 100.

$$\begin{aligned} x + y &= 120 \quad \text{Multiply by } -4 \rightarrow & -4x - 4y &= -480 \\ 10x + 4y &= 720 & \underline{10x + 4y} &= \underline{720} \\ & & 6x \quad &= 240 \quad \text{Add members} \\ & & x &= 40 \quad \text{Divide by 6} \end{aligned}$$

Using $x + y = 120$,

$$\begin{aligned} x + y &= 120 \\ (40) + y &= 120 \quad \text{Replace } x \text{ with } 40 \\ y &= 80 \end{aligned}$$

Thus, 40 cm^3 of 10% acid solution must be mixed with 80 cm^3 of 4% acid solution to obtain 120 cm^3 of 6% acid solution.

25. A metallurgist wishes to form 2,000 kilograms of an alloy that is 80% copper. He is to obtain this alloy by fusing together some alloy that is 60% pure copper with some alloy that is 85% pure copper. How many kilograms of each alloy must be used?

26. How many grams of silver that is 60% pure must be mixed together with silver that is 35% pure to obtain a mixture of 90 grams of silver that is 45% pure?

27. How many liters of a 3.5% solution and a 6% solution of acid must a chemist mix together to form 800 liters of a 4.5% acid solution?

28. How much of each substance must be mixed together if a jeweler wishes to form 16 ounces of 65% pure gold from sources that are 50% and 70% pure gold?

29. How much of an 18% salt solution must be mixed with 40 ml of a 30% salt solution to obtain a 25% salt solution?

30. How much pure salt must be mixed with 9 cubic centimeters of 20% salt solution to obtain a 40% salt solution?

31. How much pure antifreeze must be added to a 4% antifreeze mixture to obtain a 20% antifreeze mixture to fill an automobile radiator that holds 12 liters?

32. A grocer wishes to mix two candies, one selling for \$2 per pound and the other selling for \$3 per pound. How much of each candy must he mix to obtain 50 pounds of mix selling for \$2.75 per pound?

Example A boat can travel 24 miles downstream in 2 hours and 16 miles upstream in the same amount of time. What is the speed of the boat in still water and what is the speed of the current? [*Hint:* Use distance (d) = rate $(r) \times$ time (t).]

Solution Let x = speed of the boat in still water and y = speed of the current.
$x + y$ is the speed of the boat with the current downstream.
$x - y$ is the speed of the boat against the current upstream.
We set up a distance–rate–time table.

	$(d = r \cdot t)$	$\left(r = \frac{d}{t}\right)$	$\left(t = \frac{d}{r}\right)$	
	Distance	**Rate**	**Time**	
Downstream	24	$x + y$	2	$\rightarrow 24 = (x + y) \cdot 2$
Upstream	16	$x - y$	2	$\rightarrow 16 = (x - y) \cdot 2$

$$2(x + y) = 24 \qquad\qquad 2(x - y) = 16$$
$$x + y = 12 \quad \text{Divide each member by 2} \quad x - y = 8$$

We must solve the system of linear equations.

$$\begin{aligned} x + y &= 12 \\ x - y &= 8 \\ \hline 2x &= 20 \quad \text{Add members} \\ x &= 10 \end{aligned}$$

Using $x + y = 12$,

$$\begin{aligned} (10) + y &= 12 \quad \text{Replace } x \text{ with } 10 \\ y &= 2 \end{aligned}$$

The boat's speed in still water is 10 mph, and the current is 2 mph.

33. Terry travels upstream in his motorboat at top speed to town, a distance of 24 miles, in $1\frac{1}{2}$ hours. If the return trip downstream takes 1 hour, what is the top speed of Terry's motorboat and what is the speed of the stream?

34. An airplane can fly at 268 miles per hour against the wind and 380 miles per hour with the wind. What is the speed of the airplane in still air and what is the speed of the wind?

35. A jogger runs a given distance and then catches a ride back to his home by car. If the round trip of 10 miles takes 1 hour, the car travels at 40 miles per hour, and he jogs at 5 miles per hour, how long did the jogger run?

$\boxed{36.}$ Two canoeists make a 30-mile trip in 7 hours. If they paddle at a rate of 4.5 miles per hour part of the time and 4 miles per hour for the remaining time, how many hours did they travel at each rate?

37. A mother and her daughter set out at the same time from their home, jogging in opposite directions. Maintaining their normal rate, the two women are 12 miles apart after 2 hours. What is the rate of each if the daughter jogs twice as fast as her mother?

38. Two trains leave the same city at 2:00 P.M., traveling in opposite directions. If one train travels at 48 mph and the other at 60 mph, at what time will they be 594 miles apart?

39. A cyclist and a pedestrian are 40 miles apart. If they travel toward each other, they will meet in $2\frac{1}{4}$ hours. But if they travel in the same direction, the cyclist will overtake the pedestrian in 5 hours. At what rate is each traveling?

40. Two automobiles start from towns 450 miles apart and travel toward each other. They meet after 5 hours. If one automobile travels 12 miles per hour faster than the other, what is the average speed of each automobile?

41. Two automobiles are 150 miles apart. If they drive toward each other, they will meet in $1\frac{1}{2}$ hours; if they drive in the same direction, they will meet in 3 hours. What are their speeds?

Find equations for two lines, write them in standard form, and solve the system.

42. Find the equation of the line passing through points $(-1,-2)$ and $(3,4)$ and the equation of the line through points $(4,1)$ and $(2,-4)$. Find their point of intersection by solving the resulting system of linear equations.

43. Do the same as in exercise 42 for the line passing through points $(0,5)$ and $(-6,2)$ and another line passing through points $(1,1)$ and $(5,-7)$.

44. Find the point of intersection of line L_1 having slope 0 and passing through the point $(-4,-3)$ and line L_2 having slope -4 and y-intercept 2.

45. Find the point of intersection of line L_1 having slope 2 and y-intercept -6 and line L_2 having slope -3 and passing through the point $(1,2)$.

46. Find the point of intersection of line L_1 having x-intercept 3 and y-intercept -1 and line L_2 having slope 5 and passing through the point $(2,2)$.

Review exercises

Find the equation of the line, written in standard form $ax + by = c$, $a > 0$, satisfying the following conditions. See section 7–3.

1. Through points $(1,3)$ and $(-2,1)$

2. Having y-intercept $(0,-3)$ and slope $\frac{1}{2}$

3. Through the point $(2,-1)$ and parallel to the line $2x + y = 4$

In problems 4–6, evaluate the following radicals. See sections 5–1 and 5–7.

4. $-\sqrt{64}$

5. $\sqrt[3]{-27}$

6. $\sqrt{-4}$

7. Simplify the radical $\sqrt{8x^2y^3}$. See section 5–3.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8–3 ■ Systems of linear equations in three variables

Consider the equation $x + 2y - 4z = 12$, which involves three variables x, y, and z. Such an equation is called a **linear** (first-degree) **equation in three variables.** A solution of this equation is an **ordered triple** of real numbers, (x,y,z), if the resulting statement is true when the variables x, y, and z are replaced by real numbers. Then the set of all ordered triples of real numbers that satisfy the equation is the **solution set** of the equation.

To illustrate, the ordered triple $(4,0,-2)$ is a solution of the equation $x + 2y - 4z = 12$ since when we replace x with 4, y with 0, and z with -2, we obtain

$$\begin{aligned}(4) + 2(0) - 4(-2) &= 12\\ 4 + 0 + 8 &= 12\\ 12 &= 12 \quad \text{(True)}\end{aligned}$$

Other ordered triples that satisfy the equation are

$$(2,1,-2),\ (10,1,0),\ (-4,0,-4),\ \text{and}\ (-8,4,-3)$$

In this section, we will discuss the solution(s) of a system of three linear equations in three variables such as

$$\begin{aligned}3x + 2y - z &= 4\\ x - 3y + z &= -1\\ 2x + y - z &= 0\end{aligned}$$

The graph of a linear equation in three variables is a **plane.** Graphing this requires three-dimensional graphing, which is beyond the scope of this book. As with linear equations in two variables, there are a number of possible solutions.

1. The planes can intersect in one point. See figure 8–2(a).
2. The planes can have no common solutions. Two or more of the planes are parallel. See figure 8–2(b).
3. The planes can intersect in a common line. See figure 8–2(c).
4. The three planes can all be the same plane and the solutions are all points of the plane.

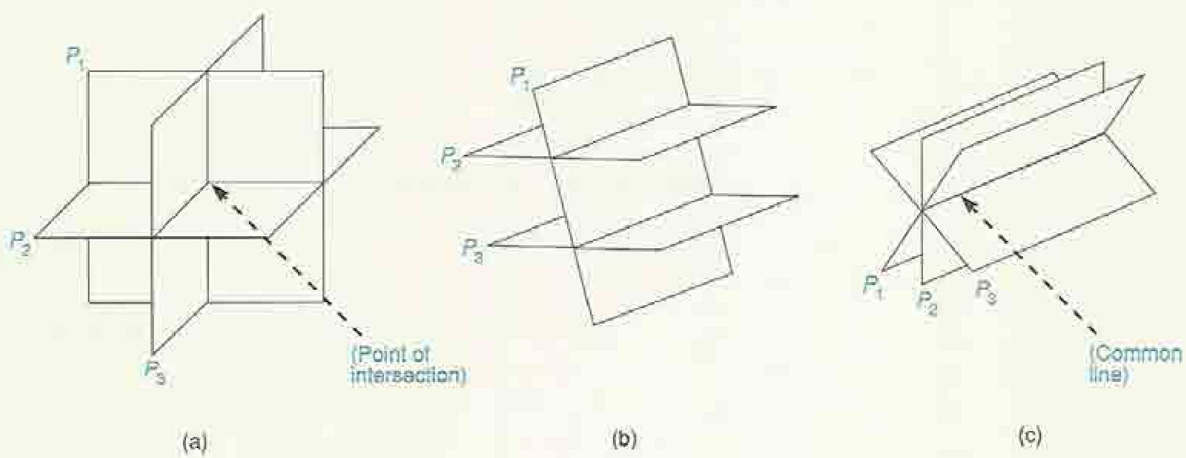

Figure 8–2

Since solving a system of three linear equations in three variables by graphing is difficult and impractical, we find the simultaneous solution by *eliminating* variables as we did with two linear equations in two variables.

Example 8–3 A

Find the solution set of each system of linear equations.

1.
$$\begin{aligned} 2x - 5y - z &= -8 \qquad (1)\\ -x + 2y + 3z &= 13 \qquad (2)\\ x + 3y - z &= 5 \qquad (3) \end{aligned}$$

The first objective is to obtain an equivalent system of two linear equations in two variables by eliminating one of the variables. Let us eliminate x from two of the equations.

To begin, use equations (2) and (3). We can eliminate x by adding the respective members of the two equations.

$$\begin{array}{rl} -x + 2y + 3z = 13 & \\ \underline{x + 3y - z = 5} & \\ 5y + 2z = 18 & (4) \quad \text{Add members} \end{array}$$

We obtain equation (4) involving the variables y and z. We must form another linear equation involving the variables y and z. Using equation (1) (the equation that has not been involved thus far) and one of the other equations, say (2), we must again eliminate x. Multiply the terms of equation (2) by 2 and add the respective members of the system.

$$\begin{array}{rll} 2x - 5y - z = -8 & (1) & \\ \underline{-2x + 4y + 6z = 26} & (2) & \text{Multiply each member of (2) by 2} \\ -y + 5z = 18 & (5) & \text{Add members} \end{array}$$

Now solve the system of linear equations in two variables formed by equations (4) and (5).

$$\begin{aligned} 5y + 2z &= 18 \qquad (4)\\ -y + 5z &= 18 \qquad (5) \end{aligned}$$

To do this, use one of the methods that we learned in section 8–1. Multiply the terms of equation (5) by 5 to obtain the system

$$\begin{array}{rl} 5y + 2z = 18 & \\ \underline{-5y + 25z = 90} & \text{Multiply (5) by 5} \\ 27z = 108 & \text{Add members} \\ z = 4 & \end{array}$$

Substitute 4 for z in equation (5) [or in equation (4)].

$$\begin{array}{rl} -y + 5(4) = 18 & \text{Replace } z \text{ with 4 in equation 5} \\ -y + 20 = 18 & \\ -y = -2 & \\ y = 2 & \text{Multiply each member by } -1 \end{array}$$

To find the third variable, x, we substitute 2 for y and 4 for z into equation (1), (2), or (3). Using equation (3),

$$\begin{aligned} x + 3y - z &= 5 \\ x + 3(2) - 4 &= 5 \qquad \text{Replace } y \text{ with 2 and } z \text{ with 4} \\ x + 6 - 4 &= 5 \\ x + 2 &= 5 \\ x &= 3 \end{aligned}$$

We have found the ordered triple (3,2,4) to be the solution of the system. The solution set is $\{(3,2,4)\}$.

2. $2x + z = 7$ (1)
$x + y = 2$ (2)
$y - z = -2$ (3)

Note that one variable is missing in each equation and it is not the same variable. Suppose we use equations (1) and (2). Since equation (3) does not contain x, we must eliminate x when working with equations (1) and (2) to obtain a system of two linear equations in the two variables y and z. Multiply the members of equation (2) by -2 and add.

$$\begin{aligned} 2x + z &= 7 \\ -2x - 2y &= -4 \qquad \text{Multiply members of (2) by } -2 \\ \hline z - 2y &= 3 \qquad (4) \quad \text{Add members} \end{aligned}$$

Now solve the system involving equations (3) and (4).

$$\begin{aligned} y - z &= -2 \qquad (3) \\ -2y + z &= 3 \qquad (4) \\ \hline -y &= 1 \qquad \text{Add members} \\ y &= -1 \qquad \text{Multiply each member by } -1 \end{aligned}$$

Replace y with -1 in equation (3).

$$\begin{aligned} y - z &= -2 \\ (-1) - z &= -2 \qquad \text{Replace } y \text{ with } -1 \\ -z &= -1 \\ z &= 1 \qquad \text{Multiply each member by } -1 \end{aligned}$$

Finally, replace z with 1 in equation (1).

$$\begin{aligned} 2x + z &= 7 \\ 2x + (1) &= 7 \qquad \text{Replace } z \text{ with 1} \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Thus the ordered triple $(3,-1,1)$ is the only solution and the solution set is $\{(3,-1,1)\}$.

Note Alternatively, we *could* have added equations (1) and (3) to eliminate z. Study each system for the easiest variable to eliminate before starting.

In general, to find the solution set of a system of three linear equations in three variables, we use the following procedure.

To solve linear systems of three equations

Step 1 Eliminate any one variable from any two of the given three equations to obtain an equation in two variables.

Step 2 Eliminate the *same* variable using the equation not yet involved and one of the other two equations. The result is another equation in the same two variables as in step 1.

Step 3 Solve the resulting system of linear equations in the two variables found in steps 1 and 2. (If this system is dependent or inconsistent, our given system is also dependent or inconsistent.)

Step 4 Substitute the values of the variables found in step 3 into any one of the three original equations to find the value of the third variable.

▶ ***Quick check*** Find the solution set of the system

$$\begin{aligned} x - y + z &= 2 \\ 2x - y + z &= 3 \\ x + 2y - 3z &= -4. \end{aligned}$$

Now we consider word problems with three unknown quantities and set up a system of three equations in three variables.

Example 8–3 B

The sum of the measures of the three angles of a triangle is 180 degrees. The middle-sized angle has measures 8° less than twice the measure of the smallest angle, and the largest angle has measures 20° less than the sum of the measure of the other two angles. Find the measure of the three angles of the triangle.

Let x = the measure of the smallest angle, y = the measure of the middle-sized angle, and z = the measure of the largest angle.
From "the sum of the measures of the three angles of a triangle is 180°,"

$$x + y + z = 180 \qquad (1)$$

From "the middle-sized angle (y) has measures 8° less than twice the measure of the smallest angle (x),"

$$y = 2x - 8 \qquad (2)$$

From "the largest angle has measures 20° less than the sum of the measure of the other two angles,"

$$z = x + y - 20 \qquad (3)$$

Thus the answer to our problem is the solution of the system of linear equations

$$\begin{aligned} x + y + z &= 180 && (1) \\ y &= 2x - 8 && (2) \\ z &= x + y - 20 && (3) \end{aligned}$$

Rewrite the system in the standard form.

$$\begin{aligned} x + y + z &= 180 && (1) \\ -2x + y &= -8 && (2) \\ -x - y + z &= -20 && (3) \end{aligned}$$

We solve the system by elimination. Since the variable z is missing from equation (2), we use equations (1) and (3) and eliminate z. Multiplying equation (3) by -1, we have the system

$$\begin{array}{rll} x + y + z = 180 & & \\ \underline{x + y - z = 20} & & \text{Multiply each member of (3) by } -1 \\ 2x + 2y \quad = 200 & (4) & \text{Add members} \end{array}$$

We now solve the system

$$\begin{array}{rll} -2x + y = -8 & (2) & \\ \underline{2x + 2y = 200} & (4) & \\ 3y = 192 & & \text{Add members} \\ y = 64 & & \end{array}$$

Using $-2x + y = -8$, replace y by 64 and solve for x.

$$\begin{array}{rl} -2x + (64) = -8 & \text{Replace } y \text{ with } 64 \\ -2x = -72 & \\ x = 36 & \end{array}$$

Replacing x with 36 and y with 64 in equation (1),

$$\begin{array}{rll} (36) + (64) + z = 180 & (1) & \text{Replace } x \text{ with } 36 \text{ and } y \text{ with } 64 \\ z = 80 & & \end{array}$$

The three angles of the triangle measure 36°, 64°, and 80°.

▶ ***Quick check*** The sum of the measures of the three angles of a triangle is 180°. The middle-sized angle has measure 2° less than twice the measure of the smallest angle, and the largest angle has measure 32° less than the sum of the measures of the other two angles. Find the measure of the three angles. ■

Mastery points

Can you

- Find the solution set of a system of three linear equations in three variables?
- Determine when a system of three linear equations in three variables is dependent or inconsistent?
- Solve a word problem by setting up a system of three linear equations in three variables and solving the problem?

Exercise 8–3

Find the solution set of the given system of linear equations. If the system is dependent or inconsistent, so state. See example 8–3 A.

Example

$$\begin{aligned} x - y + z &= 2 && (1)\\ 2x - y + z &= 3 && (2)\\ x + 2y - 3z &= -4 && (3) \end{aligned}$$

Solution Using equations (1) and (2), we eliminate z.

$$\begin{aligned} -x + y - z &= -2 && \text{Multiply each member of (1) by } -1\\ 2x - y + z &= 3 &&\\ \hline x \qquad &= 1 && \text{Add members} \end{aligned}$$

Both y and z were eliminated and $x = 1$ was obtained. Replace x with 1 in equations (1) and (3) and solve the resulting system.

$$\begin{aligned} (1) - y + z &= 2 && \text{Replace } x \text{ with } 1\\ (1) + 2y - 3z &= -4 && \text{Replace } x \text{ with } 1 \end{aligned}$$

$$\begin{aligned} -y + z &= 1 && (4) && \text{Add } -1 \text{ to each member}\\ 2y - 3z &= -5 && (5) && \text{Add } -1 \text{ to each member} \end{aligned}$$

$$\begin{aligned} -2y + 2z &= 2 && \text{Multiply each member of (4) by 2}\\ 2y - 3z &= -5 &&\\ -z &= -3 && \text{Add members}\\ z &= 3 && \text{Multiply each member by } -1 \end{aligned}$$

Using equation (1), we replace x with 1 and z with 3.

$$\begin{aligned} (1) - y + (3) &= 2 && \text{Replace } x \text{ with 1 and } z \text{ with 3}\\ 4 - y &= 2 &&\\ -y &= -2 &&\\ y &= 2 && \end{aligned}$$

The solution set is $\{(1,2,3)\}$.

1. $x + y + z = 6$, $x - 2y - z = -1$, $x + y - z = 2$
2. $x + y - z = 9$, $x + y + z = 5$, $x - y - z = 1$
3. $2x + 3y - z = 7$, $x + y + z = 6$, $3x - y - z = 6$
4. $x + y + z = 1$, $2x - y + 3z = 2$, $2x - y - z = 2$
5. $-2x + y + 4z = 3$, $x + y - 3z = 2$, $x - y + 2z = 1$
6. $3x - y + z = -8$, $4x - 2y - 3z = 3$, $2x + 3y - 2z = -1$
7. $-x + y - z = -6$, $2x + 3y - z = 1$, $x + 2y + 2z = 5$
8. $x + 2y + 3z = 5$, $-x + y - z = -6$, $2x + y + 4z = 4$
9. $-x + y + z = -3$, $3x + 9y + 5z = 5$, $x + 3y + 2z = 4$
10. $3x - 2y + 3z = 11$, $2x + 3y - 2z = -5$, $x + 4y - z = -5$
11. $6x + 4y + 2z = -1$, $3x + 2y + z = 3$, $x - 3y + z = 4$
12. $x - 2y + 3z = 4$, $3x - 3y + 4z = 5$, $2x - y + z = 1$
13. $x - 4y + z = -5$, $3x - 12y + 3z = -15$, $-2x + 8y - 2z = 10$
14. $5x - 2y + z = 6$, $-2x - 3y + 4z = -2$, $4x + 6y - 8z = 4$
15. $-5x + 3y - 2z = -13$, $4x - 2y + 5z = 13$, $2x + 4y - 3z = -9$
16. $4x + 6y + 5z = 30$, $-2x + 3y - 2z = -10$, $5x + 2y + 3z = -2$
17. $7x + 8y - 2z = -5$, $-2x + 5y + z = -3$, $5x + 14y - z = -11$

$\boxed{18.}$ $x - y = -1$
$x + z = -2$
$y - z = 2$

19. $x + 2y = 10$
$-x + 3z = -23$
$4y - z = 9$

20. $x + z = 1$
$5x + 3y = 4$
$3y - 4z = 4$

21. $2x + z = 0$
$-4x + y = 1$
$3y + z = -7$

22. $2y + z = -4$
$y = -3$
$x - 3y + 2z = 9$

23. $4x + y - 2z = 1$
$x - y = 5$
$z = -2$

24. $3x + 2y = -3$
$-2x + 5z = 38$
$4y + 3z = 12$

25. $2x - 5y = 2$
$-4x + 3z = 5$
$3y + 4z = 12$

26. $2x - y + 3z = -5$
$x + y = -4$
$2x - y + 2z = 6$

27. $x + y - z = 8$
$-x + y + z = -3$
$y + z = 5$

28. $3x + y + 5z = 3$
$5x + y - z = -3$
$8x + 2y - z = -5$

29. $2x + 8y - 2z = 6$
$3x + 12y - 3z = 9$
$-x - 4y + z = -3$

30. $x + y - 3z = 4$
$-3x + y - z = 2$
$2x + 2y - 6z = 5$

In each of the following exercises, set up a system of three linear equations in three variables and solve. See example 8–3 B.

Example The sum of the measures of the three angles of a triangle is 180°. The middle-sized angle has measure 2° less than twice the measure of the smallest angle, and the largest angle has measure 32° less than the sum of the measures of the other two angles. Find the measures of the three angles.

Solution Let x = the measure of the smallest angle, y = the measure of the middle-sized angle, and z = the measure of the largest angle.

From "the sum of the measures of the three angles of a triangle is 180°,"

$$x + y + z = 180 \qquad (1)$$

From "the middle-sized angle has measure 2° less than twice the measure of the smallest angle,"

$$y = 2x - 2 \qquad (2)$$

From "the largest angle has measure 32° less than the sum of the measures of the other two angles,"

$$z = x + y - 32 \qquad (3)$$

The system written in standard form is

$$\begin{aligned} x + y + z &= 180 \qquad (1)\\ -2x + y \phantom{{}+z} &= -2 \qquad (2)\\ x + y - z &= 32 \qquad (3) \end{aligned}$$

Using equations (1) and (3), we eliminate z.

$$\begin{aligned} x + y + z &= 180 && (1)\\ \underline{x + y - z} &= \underline{32} && (3)\\ 2x + 2y &= 212 && && \text{Add members}\\ x + y &= 106 && (4) && \text{Divide each member by 2} \end{aligned}$$

We now solve the system involving equations (2) and (4).

$$\begin{aligned} -2x + y &= -2\\ \underline{-x - y} &= \underline{-106} && \text{Multiply each member of (4) by } -1\\ -3x &= -108 && \text{Add members}\\ x &= 36 \end{aligned}$$

Using $y = 2x - 2$, replace x with 36 and solve for y.

$$\begin{aligned} y &= 2(36) - 2 && \text{Replace } x \text{ with } 36 \\ &= 72 - 2 \\ y &= 70 \end{aligned}$$

Using equation (1), we solve for z.

$$\begin{aligned} x + y + z &= 180 \\ (36) + (70) + z &= 180 && \text{Replace } x \text{ with } 36 \text{ and } y \text{ with } 70 \\ 106 + z &= 180 \\ z &= 74 \end{aligned}$$

The three angles measure 36°, 70°, and 74°.

31. The sum of the measures of the three angles of a triangle is 180°. If the largest angle is 20° more than the sum of the other two angles and the smallest angle is 67° less than the largest angle, find the measure of the three angles of the triangle.

32. The sum of the measures of the three angles of a triangle is 180°. The sum of the smallest angle and the largest angle is 120°. If the middle-sized angle has measure 30° more than the smallest angle, what is the measure of the three angles of the triangle?

33. The perimeter of a triangular-shaped garden is 122 meters. If the length of the longest side is equal to the sum of the lengths of the other two sides, and twice the length of the shortest side is 11 meters less than the length of the longest side, find the lengths of the three sides. (*Note:* Perimeter is the distance around the triangle.)

34. The longest side of a triangle is twice the length of the shortest side, and the middle-length side is 9 inches longer than the shortest side. If the perimeter is 65 inches, what are the lengths of the three sides?

35. Tickets for a Harry Belafonte concert have three prices, "expensive," "middle-priced," and "cheapest." The "middle-priced" tickets cost \$4 more than the "cheapest," and the "expensive" tickets cost \$6 more than the "middle-priced" tickets. If the "expensive" tickets cost \$1 less than twice the "cheapest" tickets, find the price of each kind of ticket.

36. A used-car salesman must sell a quota of cars before receiving a bonus. The cars are placed in three different price categories—A, B, and C. He must sell two more at price B than at price A and three times as many at price C as at price A. If the number at price C is one more than twice the number at price B, find the number of each category of car that he must sell.

37. Bill has a special stamp collection that is worth approximately \$151,000 on the market. The stamps are separated into three different approximate price categories—\$750, \$1,500, and \$25,000 per stamp. If the number of \$750 stamps is four times the number of \$25,000 stamps and the number of \$1,500 stamps is ten more than the number of \$750 stamps, how many of each kind does Bill's collection contain?

38. Erin has a collection of pennies, nickels, and dimes in her piggy bank. She has twice as many pennies as dimes and eight more nickels than dimes. If she has \$2.44 altogether, how many of each coin does she have?

39. Jay has 33 bills in denominations of fives, tens, and twenties. If he has \$360 total and the number of five-dollar bills is two more than the number of twenty-dollar bills, how many of each denomination does he have?

40. Find the values of a, b, and c so that the points $(0,5)$, $(-1,2)$, and $(2,17)$ lie on the graph of $y = ax^2 + bx + c$. (*Hint:* Substitute the coordinates of each point into the equation to obtain three linear equations in variables a, b, and c.)

41. Find the values of a, b, and c so that the points $(1,-2)$, $(0,2)$, and $(-2,13)$ lie on the graph of $y = ax^2 + bx + c$.

42. Find the values of a, b, and c so that the points $(2,-8)$, $(-1,-2)$, and $(3,-22)$ lie on the graph of $y = ax^2 + bx + c$.

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Review exercises

1. Write 0.000247 in scientific notation. See section 3–3.
2. Subtract $\dfrac{4y - 3}{y^2 - 2y - 3} - \dfrac{2y + 5}{y^2 - 1}$. See section 4–3.
3. Evaluate $3a - 2b + c$ when $a = 4$, $b = -5$, and $c = -6$. See section 1–5.
4. Graph the linear inequality $2x - y \leq 4$. See section 7–4.
5. Find the distance from the point $(1,-3)$ to the point $(-6,1)$ in the plane. What are the coordinates of the midpoint of the line segment with these endpoints? See section 7–2.
6. Solve the quadratic-type equation $3x^4 - x^2 - 4 = 0$. See section 6–6.

8–4 ■ Determinants

In sections 8–1 and 8–3, we solved systems of linear equations in two and three variables by using algebraic methods involving the elimination of a variable and by substituting an expression for one of the variables. We now consider another method for solving these systems by using **determinants.** A determinant is the number associated with an array of numbers called a *matrix.*

The rectangular array of numbers shown below is called a "three by two" matrix (denoted 3×2) because there are three rows and two columns in the matrix.

$$\begin{array}{cl} \begin{bmatrix} 3 & 4 \\ -1 & 0 \\ 2 & -3 \end{bmatrix} & \text{3 rows} \\ \text{2 columns} & \end{array}$$

Note The array of numbers making up a matrix is enclosed within a set of brackets.

When the matrix has the same number of rows as columns, the array is called a *square matrix.* An example of a 2×2 square matrix and a 3×3 square matrix is shown.

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{bmatrix} \\ 2 \times 2 & 3 \times 3 \end{array}$$

Associated with every square matrix having real number entries is a real number called its *determinant.* To denote the determinant, we enclose the array between two vertical lines. The determinant of the

$$2 \times 2 \text{ matrix } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ is written } \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$3 \times 3 \text{ matrix } \begin{bmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{bmatrix} \text{ is written } \begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

The value of a 2×2 determinant is found by the following:

Value of a 2×2 determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Example 8–4 A

Find the value of each determinant.

1. $\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(-2) = 4 + 6 = 10$

2. $\begin{vmatrix} 0 & 3 \\ -4 & -1 \end{vmatrix} = (0)(-1) - (-4)(3) = 0 + 12 = 12$

▶ ***Quick check*** Find the value of the determinant $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$.

The value of the determinant of a 3×3 matrix is found by rewriting the determinant in terms of 2×2 determinants, which we call *minors*. We now define the minor of an element, or entry, of a 3×3 determinant.

Definition of a minor

The **minor** of an element of a 3×3 determinant is defined to be the 2×2 determinant that remains after the row and column in which the element appears have been deleted.

Given the determinant $\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix}$, to find the minor of

1. 4 in the first column, eliminate the row and column that contain 4.

$$\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

The minor of 4 is $\begin{vmatrix} 7 & 4 \\ -1 & 6 \end{vmatrix}$.

2. 2, eliminate the row and column that contain 2.

$$\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

The minor of 2 is $\begin{vmatrix} -7 & 5 \\ -1 & 6 \end{vmatrix}$.

3. 0, eliminate the row and column that contain 0.

$$\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix}$$

The minor of 0 is $\begin{vmatrix} -7 & 5 \\ 7 & 4 \end{vmatrix}$.

The value of this 3×3 determinant is found by *expanding by minors* about the elements of one row or one column. To do this, we multiply each element of that row (or column) by its minor using the following sign pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Thus, using the elements of the first column in the expansion,

$$\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix} = +4\begin{vmatrix} 7 & 4 \\ -1 & 6 \end{vmatrix} - 2\begin{vmatrix} -7 & 5 \\ -1 & 6 \end{vmatrix} + 0\begin{vmatrix} -7 & 5 \\ 7 & 4 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{aligned} &= 4[(7)(6) - (-1)(4)] - 2[(-7)(6) - (-1)(5)] \\ &\quad + 0[(-7)(4) - (7)(5)] \\ &= 4(42 + 4) - 2(-42 + 5) + 0(-28 - 35) \\ &= 4(46) - 2(-37) + 0(-63) \\ &= 184 + 74 + 0 \\ &= 258 \end{aligned}$$

The value of the 3×3 determinant is 258.

Note Had we chosen to evaluate the determinant about the elements of the *second row*,

$$\begin{vmatrix} 4 & -7 & 5 \\ 2 & 7 & 4 \\ 0 & -1 & 6 \end{vmatrix} = -2\begin{vmatrix} -7 & 5 \\ -1 & 6 \end{vmatrix} + 7\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} - 4\begin{vmatrix} 4 & -7 \\ 0 & -1 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$\begin{aligned} &= -2(-42 + 5) + 7(24 - 0) - 4(-4 - 0) \\ &= -2(-37) + 7(24) - 4(-4) \\ &= 74 + 168 + 16 \\ &= 258 \qquad \text{(The same result)} \end{aligned}$$

In general, to expand a 3×3 determinant by minors about the elements of the first column, we use the following procedure.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Example 8–4 B

Expand the determinant by minors about the first column.

$$\begin{aligned}\begin{vmatrix} 2 & -3 & -1 \\ 1 & 2 & 3 \\ -4 & -2 & 1 \end{vmatrix} &= +2\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 1\begin{vmatrix} -3 & -1 \\ -2 & 1 \end{vmatrix} + (-4)\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} \\ &= 2(2+6) - 1(-3-2) + (-4)(-9+2) \\ &= 2(8) - 1(-5) + (-4)(-7) \\ &= 16 + 5 + 28 \\ &= 49\end{aligned}$$

The value of the 3×3 determinant is 49.

▶ ***Quick check*** Expand the determinant by minors about the first column.

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & -2 \\ -1 & 2 & -3 \end{vmatrix}$$

The expansion of higher-order determinants by minors can be accomplished in the same way as with third-order determinants. The pattern of alternating signs in the sign array extends to higher-order determinants. The determinants that are the minors in each expansion will be of order one less than the order of the original determinant. The following is the sign array of a 4×4 determinant.

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

Example 8–4 C

Expand the determinant by minors about the first row.

$$\begin{vmatrix} -1 & -2 & 3 & 2 \\ 0 & 1 & 4 & -2 \\ 3 & -1 & 4 & 0 \\ 2 & 1 & 0 & 3 \end{vmatrix}$$

$$= +(-1)\begin{vmatrix} 1 & 4 & -2 \\ -1 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 0 & 4 & -2 \\ 3 & 4 & 0 \\ 2 & 0 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 & -2 \\ 3 & -1 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$- 2\begin{vmatrix} 0 & 1 & 4 \\ 3 & -1 & 4 \\ 2 & 1 & 0 \end{vmatrix}$$

Evaluating each 3×3 determinant by expanding about three 2×2 minors, we obtain the following.

$$\begin{aligned} &= -1(32) + 2(-20) + 3(-19) - 2(28) \\ &= -185 \end{aligned}$$

Mastery points

Can you

- Evaluate 2 × 2, 3 × 3, and 4 × 4 determinants?

Exercise 8–4

Evaluate the following determinants. See example 8–4 A.

Example $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix}$

Solution $\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = (3)(2) - (-1)(4) = 6 + 4 = 10$

1. $\begin{vmatrix} 1 & -3 \\ 2 & 5 \end{vmatrix}$

2. $\begin{vmatrix} 4 & 1 \\ 2 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 4 & -2 \\ -6 & -3 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 5 \\ 4 & 10 \end{vmatrix}$

5. $\begin{vmatrix} 0 & -5 \\ 1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 4 & 0 \\ 2 & -7 \end{vmatrix}$

7. $\begin{vmatrix} 5 & 4 \\ 0 & -7 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}$

Evaluate the following determinants using expansion by minors about any column or row. See example 8–4 B.

Example $\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & -2 \\ -1 & 2 & -3 \end{vmatrix}$

Solution Using the first column,

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 4 & -2 \\ -1 & 2 & -3 \end{vmatrix} = +3\begin{vmatrix} 4 & -2 \\ 2 & -3 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 2 & -3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix}$$
$$= 3(-12 + 4) - 0(-6 - 2) - 1(-4 - 4)$$
$$= 3(-8) - 0(-8) - 1(-8) = -24 - 0 + 8$$
$$= -16.$$

9. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix}$

10. $\begin{vmatrix} -2 & 1 & 3 \\ 1 & 0 & 4 \\ -1 & 4 & 3 \end{vmatrix}$

11. $\begin{vmatrix} -3 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & 4 & 5 \end{vmatrix}$

12. $\begin{vmatrix} 3 & 0 & 1 \\ 2 & 0 & 4 \\ 3 & 0 & -1 \end{vmatrix}$

13. $\begin{vmatrix} 3 & -1 & 2 \\ 0 & 0 & 0 \\ 4 & 3 & 1 \end{vmatrix}$

14. $\begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

15. $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$

16. $\begin{vmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 3 & -3 & 3 \end{vmatrix}$

17. $\begin{vmatrix} -1 & 4 & 0 \\ -2 & 1 & -3 \\ -3 & 7 & 2 \end{vmatrix}$

18. $\begin{vmatrix} 0 & 1 & 7 \\ 3 & 0 & 2 \\ 4 & 0 & -1 \end{vmatrix}$

19. $\begin{vmatrix} -4 & 0 & 1 \\ 2 & 0 & 3 \\ -5 & -1 & -2 \end{vmatrix}$

20. $\begin{vmatrix} 5 & 10 & 15 \\ 1 & -1 & 0 \\ 2 & 1 & 0 \end{vmatrix}$

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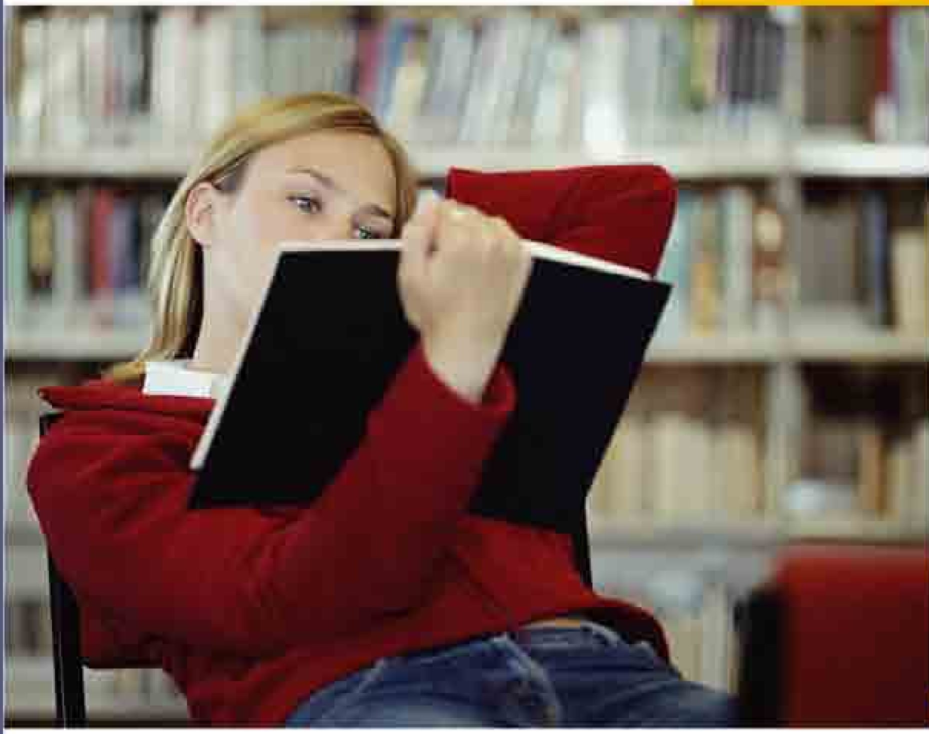

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21. $\begin{vmatrix} 2 & 2 & -2 \\ 3 & -3 & 3 \\ -1 & 1 & 1 \end{vmatrix}$ **22.** $\begin{vmatrix} 0 & -1 & 2 \\ 1 & 0 & 2 \\ -5 & 6 & 0 \end{vmatrix}$ **23.** $\begin{vmatrix} a & 0 & a \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$

24. $\begin{vmatrix} 0 & x & 0 \\ 0 & 0 & x \\ x & 0 & 0 \end{vmatrix}$ **25.** $\begin{vmatrix} x & y & 0 \\ 0 & x & y \\ y & x & 0 \end{vmatrix}$ **26.** $\begin{vmatrix} a & b & 0 \\ b & 0 & a \\ 0 & a & b \end{vmatrix}$

Expand and evaluate each determinant about the first row using the sign array $+ - + -$. See example 8–4 C.

27. $\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 0 \end{vmatrix}$ **28.** $\begin{vmatrix} 0 & 5 & 3 & 1 \\ 2 & 0 & 4 & -2 \\ -1 & -1 & 0 & 1 \\ 2 & 1 & 3 & 0 \end{vmatrix}$

29. $\begin{vmatrix} 4 & 0 & 1 & 0 \\ -5 & -1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 2 & 0 & -3 & 1 \end{vmatrix}$ **30.** $\begin{vmatrix} 0 & 0 & 1 & 4 \\ -5 & 6 & 0 & -3 \\ 2 & 0 & 1 & 0 \\ 4 & 2 & 0 & 7 \end{vmatrix}$

Review exercises

1. Solve the system of linear equations by elimination. See section 8–1. $\begin{aligned} 3x - y &= -8 \\ 2x + 3y &= 2 \end{aligned}$

Graph the following equations. See section 7–1.

2. $y = 3x - 2$ **3.** $x = -2$ **4.** $2x + 3y = -6$

5. Given $P(x) = x^2 - 6x + 9$, find $P(-1)$. See section 1–5.

6. Find the solution set of the equation $y - 3\sqrt{y} + 2 = 0$. See section 6–6.

8–5 ■ Solutions of systems of linear equations by determinants

Determinants can be used to solve systems of linear equations. Consider the system of two linear equations in two variables

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned} \quad \text{(Written in standard form)}$$

Suppose we solve the system of equations by elimination. To eliminate x multiplying each term of the first equation by $-a_2$ and each term of the second equation by a_1, we obtain the equivalent system

$$\begin{aligned} -a_1a_2x - a_2b_1y &= -a_2c_1 \\ a_1a_2x + a_1b_2y &= a_1c_2 \end{aligned}$$

Adding the terms, the resulting equation in y is given by

$$a_1b_2y - a_2b_1y = a_1c_2 - a_2c_1$$

Factoring y from the terms in the left member, we obtain

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$$

and dividing each member by $a_1b_2 - a_2b_1$, we have

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \qquad (a_1b_2 - a_2b_1 \neq 0)$$

In like fashion, if we solve the system for x, we obtain

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1} \qquad (a_1b_2 - a_2b_1 \neq 0)$$

Now by definition,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \text{ (which we denote by } D\text{)}$$

$$\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = b_2c_1 - b_1c_2 \text{ (which we denote by } D_x\text{)}$$

$$\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1 \text{ (which we denote by } D_y\text{)}$$

Note
1. D contains the coefficients of the variables in order as they appear in the equations.
2. D_x contains the coefficients of the variables replacing the x-coefficients with the constants in the right member.
3. D_y contains the coefficients of the variables replacing the y-coefficients with the constants in the right member.

To solve this system of equations for x and for y, we obtain the following determinant ratios used to solve a system of linear equations by determinants (called *Cramer's Rule*).

Cramer's Rule for 2 × 2 linear systems

Given the system of linear equations
$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$
then

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

where $D = a_1b_2 - a_2b_1$, $D \neq 0$.

Example 8–5 A

Use Cramer's Rule to find the solution set of each system of linear equations.

1. $2x - 3y = 4$
$x + 4y = -1$

By Cramer's Rule, $x = \dfrac{D_x}{D}$ and $y = \dfrac{D_y}{D}$. To find D, we use the coefficients of the variables.

$$D = \begin{vmatrix} 2 & -3 \\ 1 & 4 \end{vmatrix} = 2(4) - 1(-3) = 8 + 3 = 11$$

To find D_x, replace the column $\begin{matrix}2\\1\end{matrix}$ of D by the constants $\begin{matrix}4\\-1\end{matrix}$.

$$D_x = \begin{vmatrix} 4 & -3 \\ -1 & 4 \end{vmatrix} = 4(4) - (-1)(-3) = 16 - 3 = 13$$

To find D_y, replace the column $\begin{matrix}-3\\4\end{matrix}$ of D by the constants $\begin{matrix}4\\-1\end{matrix}$.

$$D_y = \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} = 2(-1) - 1(4) = -2 - 4 = -6$$

By Cramer's Rule,

$$x = \frac{D_x}{D} = \frac{13}{11}$$
$$y = \frac{D_y}{D} = \frac{-6}{11} = -\frac{6}{11}$$

The solution set of the system is $\left\{\left(\frac{13}{11}, -\frac{6}{11}\right)\right\}$.

Note If $D = 0$, the system of equations is then either inconsistent or dependent. When

$$D = 0 \text{ and } D_y \neq 0,\ D_x \neq 0$$

the system of equations is *inconsistent*. If

$$D = 0 \text{ and } D_x = D_y = 0$$

the system of equations is *dependent*.

2. $4x - y = 3 \qquad (1)$

$2x - \frac{y}{2} = 0 \qquad (2)$

We first clear the denominator of equation (2).

$$2x - \frac{y}{2} = 0$$
$$2 \cdot 2x - 2 \cdot \frac{y}{2} = 2 \cdot 0 \qquad \text{Multiply each member by 2}$$
$$4x - y = 0$$

We now solve the system of linear equations.

$$4x - y = 3$$
$$4x - y = 0$$

Using determinants,

$$D = \begin{vmatrix} 4 & -1 \\ 4 & -1 \end{vmatrix} = (-4) - (-4) = 0$$

$$D_x = \begin{vmatrix} 3 & -1 \\ 0 & -1 \end{vmatrix} = (-3) - (0) = -3$$

$$D_y = \begin{vmatrix} 4 & 3 \\ 4 & 0 \end{vmatrix} = (0) - (12) = -12$$

Since $D = 0$, $D_x \neq 0$, and $D_y \neq 0$, the system is inconsistent and the solution set is $\emptyset$.

▶ ***Quick check*** Use Cramer's Rule to find the solution set of $3x - 4y = 1$
$x + 2y = -4$. ■

Now we consider a system of three linear equations in three variables. The procedure is similar to that used to solve a system of two linear equations in two variables.

Cramer's Rule for 3 × 3 linear systems

Given the system of linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

(Written in standard form)

we define

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Determinant of coefficients

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Constants replacing x-coefficients

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

Constants replacing y-coefficients

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Constants replacing z-coefficients

Then $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{D}$, $D \neq 0$.

■ Example 8–5 B

Use Cramer's Rule to find the solution set of the system of linear equations.

$$\begin{aligned} 2x - 2y + z &= 0 \\ x + 5y - 7z &= 3 \\ x - y - 3z &= -7 \end{aligned}$$

We first evaluate the denominator D by minors about the first column.

$$\begin{aligned} D = \begin{vmatrix} 2 & -2 & 1 \\ 1 & 5 & -7 \\ 1 & -1 & -3 \end{vmatrix} &= 2\begin{vmatrix} 5 & -7 \\ -1 & -3 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ -1 & -3 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ 5 & -7 \end{vmatrix} \\ &= 2(-15 - 7) - 1(6 + 1) + 1(14 - 5) \\ &= -42 \end{aligned}$$

$$D_x = \begin{vmatrix} 0 & -2 & 1 \\ 3 & 5 & -7 \\ -7 & -1 & -3 \end{vmatrix} = 0\begin{vmatrix} 5 & -7 \\ -1 & -3 \end{vmatrix} - 3\begin{vmatrix} -2 & 1 \\ -1 & -3 \end{vmatrix} + (-7)\begin{vmatrix} -2 & 1 \\ 5 & -7 \end{vmatrix}$$
$$= 0(-15 - 7) - 3(6 + 1) - 7(14 - 5)$$
$$= -84$$

$$D_y = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & -7 \\ 1 & -7 & -3 \end{vmatrix} = 2\begin{vmatrix} 3 & -7 \\ -7 & -3 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ -7 & -3 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 3 & -7 \end{vmatrix}$$
$$= 2(-9 - 49) - 1(0 + 7) + 1(0 - 3)$$
$$= -126$$

$$D_z = \begin{vmatrix} 2 & -2 & 0 \\ 1 & 5 & 3 \\ 1 & -1 & -7 \end{vmatrix} = 2\begin{vmatrix} 5 & 3 \\ -1 & -7 \end{vmatrix} - 1\begin{vmatrix} -2 & 0 \\ -1 & -7 \end{vmatrix} + 1\begin{vmatrix} -2 & 0 \\ 5 & 3 \end{vmatrix}$$
$$= 2(-35 + 3) - 1(14 - 0) + 1(-6 - 0)$$
$$= -84$$

$$x = \frac{D_x}{D} \qquad y = \frac{D_y}{D} \qquad z = \frac{D_z}{D}$$
$$x = \frac{-84}{-42} = 2, \quad y = \frac{-126}{-42} = 3, \quad z = \frac{-84}{-42} = 2$$

The solution set is $\{(2,3,2)\}$. *Be sure to check your solution.*

▶ ***Quick check*** Use Cramer's Rule to find the solution set of the system
$x - 2y + z = 0$
$2x - y + 3z = 4$
$x - y - 2z = -2.$

Mastery points

Can you

- Solve a system of linear equations in two or three variables using Cramer's Rule?

Exercise 8–5

Using Cramer's Rule, find the solution set of the following systems of linear equations. See example 8–5 A.

Example $3x - 4y = 1$
$x + 2y = -4$

Solution $D = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix} = (3)(2) - (1)(-4) = 6 + 4 = 10$

Replace $\begin{matrix} 3 \\ 1 \end{matrix}$ with the constants $\begin{matrix} 1 \\ -4 \end{matrix}$ to find D_x.

$D_x = \begin{vmatrix} 1 & -4 \\ -4 & 2 \end{vmatrix} = (1)(2) - (-4)(-4) = 2 - 16 = -14$

Replace $\begin{matrix} -4 \\ 2 \end{matrix}$ with constants $\begin{matrix} 1 \\ -4 \end{matrix}$ to find D_y.

$D_y = \begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix} = (3)(-4) - (1)(1) = -12 - 1 = -13$

By Cramer's Rule,

$x = \dfrac{D_x}{D} = \dfrac{-14}{10} = -\dfrac{7}{5}$

$y = \dfrac{D_y}{D} = \dfrac{-13}{10} = -\dfrac{13}{10}$

The solution set is $\left\{\left(-\dfrac{7}{5}, -\dfrac{13}{10}\right)\right\}$.

1. $x - y = 2$
   $2x + y = 3$
2. $3x + y = 8$
   $x - 2y = -1$
3. $3x + 5y = 6$
   $2x - 3y = -4$
4. $4x - y = 3$
   $8x - 2y = 1$
5. $10x - 2y = -3$
   $5x - y = 0$
6. $x + 2y = 7$
   $3x + 6y = 21$
7. $-4x - 6y = 2$
   $2x + 3y = -1$
8. $6x - 2y = 7$
   $x - 6y = 11$
9. $-3x - y = 1$
   $4x + 5y = -5$
10. $2x - 3y = 1$
    $-3x + 5y = 2$
11. $6x - 2y = 7$
    $2y = 8$
12. $x - 7y = -3$
    $-y = 8$
13. $4x - 5y = -33$
    $-x + 3y = 17$
14. $3x + 15y = -1$
    $x + 5y = 6$
15. $\frac{1}{3}x - \frac{3}{2}y = 6$
    $-\frac{2}{3}x + \frac{1}{2}y = -7$
16. $-\frac{1}{4}x + 3y = -8$
    $x + \frac{2}{3}y = -6$

See example 8–5 B.

Example
$$\begin{aligned} x - 2y + z &= 0 \\ 2x - y + 3z &= 4 \\ x - y - 2z &= -2 \end{aligned}$$

Solution Using minors about column 1,

$$D = \begin{vmatrix} 1 & -2 & 1 \\ 2 & -1 & 3 \\ 1 & -1 & -2 \end{vmatrix} = 1\begin{vmatrix} -1 & 3 \\ -1 & -2 \end{vmatrix} - 2\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix}$$
$$= 1(2 + 3) - 2(4 + 1) + 1(-6 + 1)$$
$$= -10$$

$$D_x = \begin{vmatrix} 0 & -2 & 1 \\ 4 & -1 & 3 \\ -2 & -1 & -2 \end{vmatrix} = 0\begin{vmatrix} -1 & 3 \\ -1 & -2 \end{vmatrix} - 4\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix} + (-2)\begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix}$$
$$= 0 - 4(4 + 1) - 2(-6 + 1)$$
$$= -10$$

$$D_y = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 4 & 3 \\ 1 & -2 & -2 \end{vmatrix} = 1\begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix}$$
$$= 1(-8 + 6) - 2(0 + 2) + 1(0 - 4)$$
$$= -10$$

$$D_z = \begin{vmatrix} 1 & -2 & 0 \\ 2 & -1 & 4 \\ 1 & -1 & -2 \end{vmatrix} = 1\begin{vmatrix} -1 & 4 \\ -1 & -2 \end{vmatrix} - 2\begin{vmatrix} -2 & 0 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -2 & 0 \\ -1 & 4 \end{vmatrix}$$
$$= 1(2 + 4) - 2(4 - 0) + 1(-8 - 0)$$
$$= -10$$

$$x = \frac{D_x}{D} = \frac{-10}{-10} = 1, \quad y = \frac{D_y}{D} = \frac{-10}{-10} = 1, \quad z = \frac{D_z}{D} = \frac{-10}{-10} = 1$$

The solution set is $\{(1,1,1)\}$.

$\boxed{17.}$ $\begin{aligned} 4x - y + 2z &= 0 \\ 2x + y + z &= 3 \\ 3x - y + z &= -2 \end{aligned}$

18. $\begin{aligned} x - y - z &= 6 \\ 2x + 3y - z &= 1 \\ x + 2y + 2z &= 5 \end{aligned}$

19. $\begin{aligned} x - 2y - z &= 0 \\ 2x - y + z &= 0 \\ 4x + 2y - 2z &= 0 \end{aligned}$

20. $\begin{aligned} 2x + y - z &= 0 \\ x - 3y + 3z &= 0 \\ -3x + 2y - 2z &= 0 \end{aligned}$

21. $\begin{aligned} 3x - y - 6z &= 5 \\ -6x + 2y - 2z &= -4 \\ 3x - y + z &= 2 \end{aligned}$

22. $\begin{aligned} 6x - 9y + 12z &= 24 \\ -4x + 6y - 8z &= -16 \\ 2x - 3y + 4z &= 8 \end{aligned}$

23. $\begin{aligned} 2x - y + 3z &= 5 \\ 3x - y + 2z &= 10 \\ 3x - 2y + 7z &= 3 \end{aligned}$

24. $\begin{aligned} 4x + 2y - 3z &= 2 \\ 6x + y - 2z &= 0 \\ 2x + 3y - z &= 0 \end{aligned}$

25. $\begin{aligned} 3x + 2y + 4z &= 3 \\ 3x - y - 2z &= 0 \\ 3x + y + 2z &= 2 \end{aligned}$

26. $\begin{aligned} 2x - 7y + 3z &= 1 \\ 4x - y - 6z &= -6 \\ -2x + 5y - 3z &= -1 \end{aligned}$

27. $\begin{aligned} 3x - 2y + 4z &= -6 \\ 2x - 2y + 4z &= -1 \\ x - y + 2z &= 4 \end{aligned}$

28. $\begin{aligned} 2x + 3y + 4z &= 13 \\ -x - y + 5z &= 4 \\ x + y - 7z &= -6 \end{aligned}$

29. $\begin{aligned} -3x - y + 4z &= 1 \\ 5x - y + 2z &= 10 \\ 7x + 2y - 2z &= -4 \end{aligned}$

$\boxed{30.}$ $\begin{aligned} x + y &= 1 \\ y + 2z &= -2 \\ 2x - z &= 0 \end{aligned}$

31. $\begin{aligned} 2y + z &= 6 \\ 3x + 4z &= 14 \\ 3x - y &= 4 \end{aligned}$

32. $x - 4y + z = -4$
$4x + 2y - 3z = 6$
$-x + 2z = 2$

33. $x - 2y + z = -2$
$3x + y = 7$
$2x - z = 0$

34. $y + 4z = 6$
$4x + z = 0$
$5y - z = 9$

35. $3x + 4y - 2z = -25$
$2x - y + 3z = -5$
$-x + z = 6$

36. $x - y + z = -9$
$3x + 4y = 6$
$2y - z = 10$

In exercises 37–40, select two variables to represent two unknowns, set up a system of two linear equations in two variables and solve the resulting system using Cramer's Rule.

37. The length of a rectangular garden is two feet longer than twice the width. Find the dimensions of the garden if the perimeter is 46 feet.

38. The sum of two numbers is 102. If their difference is 48, find the two numbers.

39. A donut shop sells 5 cream-filled and 7 jelly-filled donuts for \$4.04 and 3 cream-filled and 9 jelly-filled donuts for \$3.96. Find the cost of a single cream-filled and a single jelly-filled donut.

40. A bank teller receives 145 in \$5 and \$10 bills. If they are \$1,190 in value, how many of each bill did she receive?

In exercises 41–43, select three variables to represent three unknowns, set up a system of three linear equations in three variables and solve the resulting system using Cramer's Rule.

41. A bank teller has \$565 in \$5, \$10, and \$20 bills. The number of \$10 bills is twice the number of \$5 and \$20 bills put together. How many of each bill are there if there are 48 bills in all?

42. The sum of three numbers is 36. If the second number is 1 more than the first number and the last number is $1\frac{1}{2}$ times the first, find the three numbers.

43. Sarah has a collection of 23 stuffed animals—elephants, bears, and dogs. She has four times as many bears as she has elephants and the number of dogs is two more than twice the number of elephants. How many of each animal does she have?

Review exercises

1. Given the linear equation $3x + y = 6$, complete the given ordered pairs and graph the equation. See section 7–1. $(-2, \;)$; $(-1, \;)$; $(0, \;)$; $(1, \;)$; $(2, \;)$

2. Find the solution set of the inequality $2x^2 + 5x - 3 > 0$. See section 6–7.

3. Find the solution set of the quadratic equation $3x^2 - 16x = -5$. See section 6–3.

4. Find the solution set of the equation $\frac{3}{x} - x = 2$. See section 6–1.

Perform the indicated operations. See sections 5–6 and 5–7.

5. $(2\sqrt{3} - 5)^2$

6. $(4\sqrt{2} - 3)(4\sqrt{2} + 3)$

7. i^{13}

8–6 ■ Solving systems of linear equations by the augmented matrix method

We have learned several methods for solving systems of linear equations. Now we will consider a method for solving these systems using the *matrix* that was defined in section 8–4. This method is easily adapted to a computer.

With every system of equations, we can associate a matrix that consists of the coefficients and constant terms. To illustrate, consider the system

$$\begin{aligned} 6x + 5y &= -3 \\ 2x + y &= -7 \end{aligned}$$

with which we can associate the matrix

$$\left[\begin{array}{cc|c} 6 & 5 & -3 \\ 2 & 1 & -7 \end{array}\right]$$

We call this the **augmented matrix** of the system. The coefficients of the variables are placed to the left of the vertical bar and the constants to the right. We now operate on the rows of the augmented matrix just as we did with the equations of the system. Using the following **elementary row operations,** we produce new matrices that lead to systems containing solutions of the original system.

Elementary row operations

1. Any two rows of the augmented matrix can be interchanged.
2. Any row can be multiplied by a nonzero constant.
3. Any row of the augmented matrix can be changed by adding a nonzero multiple of another row to that row.

Note We used these same operations when solving a system by the elimination (addition) method.

Use row operations to rewrite the matrix until it is a system whose solution is easy to find. The object is to obtain the first column $\begin{matrix}1\\0\end{matrix}$ and the second column $\begin{matrix}\text{constant}\\1\end{matrix}$ that now represent the coefficients of the variables x and y. We can then easily solve the resulting system.

■ Example 8–6 A

Find the solution set of the following linear system using the augmented matrix method.

$$\begin{aligned} 6x + 5y &= -3 \\ 2x + y &= -7 \end{aligned}$$

The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 6 & 5 & -3 \\ 2 & 1 & -7 \end{array}\right]$$

To start with, we want to make sure that there is a 1 in the first row, first column position. We multiply the first row by $\frac{1}{6}$ to obtain the matrix

$$\left[\begin{array}{cc|c} 1 & \frac{5}{6} & -\frac{1}{2} \\ 2 & 1 & -7 \end{array}\right]$$

Next, we must get 0s in every position below the first position. To get 0 in the second row and the first column, use the third row operation: add to the numbers in the second row the results of multiplying the numbers in the first row by -2. We obtain

$$\left[\begin{array}{cc|c} 1 & \frac{5}{6} & -\frac{1}{2} \\ 2 + 1(-2) & 1 + \frac{5}{6}(-2) & -7 + \left(-\frac{1}{2}\right)(-2) \end{array}\right]$$

Original number + (-2) times number from the first row

$$\left[\begin{array}{cc|c} 1 & \frac{5}{6} & -\frac{1}{2} \\ 0 & -\frac{2}{3} & -6 \end{array}\right]$$

We now have 1 in the first row, first column position and 0s in every position below that. Now we go to the second column and obtain 1 in the second row, second column position. To get this, use the second row operation and multiply the second row by $-\frac{3}{2}$.

$$\left[\begin{array}{cc|c} 1 & \frac{5}{6} & -\frac{1}{2} \\ 0 & 1 & 9 \end{array}\right]$$

This augmented matrix yields the system

$$x + \frac{5}{6}y = -\frac{1}{2}$$
$$0x + 1y = 9 \text{ (or } y = 9)$$

Thus $y = 9$ and substituting 9 for y in the equation $x + \frac{5}{6}y = -\frac{1}{2}$, we obtain

$$x + \frac{5}{6}(9) = -\frac{1}{2} \qquad \text{Replace } y \text{ with } 9$$
$$x + \frac{15}{2} = -\frac{1}{2}$$
$$x = -\frac{1}{2} - \frac{15}{2}$$
$$x = -8$$

The solution set of the system is $\{(-8,9)\}$.

To solve systems with three equations, use row operations to get 1's down the diagonal from left to right and 0s below each 1. The following example demonstrates this method.

Example 8–6 B

Find the solution set of the following system using the augmented matrix method.

$$\begin{aligned} 2x - y + 2z &= -8 \\ x + 2y - 3z &= 9 \\ 3x - y - 4z &= 3 \end{aligned}$$

Write the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 2 & -1 & 2 & -8 \\ 1 & 2 & -3 & 9 \\ 3 & -1 & -4 & 3 \end{array}\right]$$

To obtain 1 in the first row and the first column, we interchange the first and second rows.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 2 & -1 & 2 & -8 \\ 3 & -1 & -4 & 3 \end{array}\right]$$

We must now get 0s in the first column below the first row. Add to the second row the results of multiplying the first row by -2.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 3 & -1 & -4 & 3 \end{array}\right]$$

Add to the third row the results of multiplying the first row by -3.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & -7 & 5 & -24 \end{array}\right]$$

We must now obtain a 0 in the third row and the second column. Add to the third row the results of multiplying the second row by $-\frac{7}{5}$.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & -\frac{31}{5} & \frac{62}{5} \end{array}\right]$$

Multiply the third row by $-\frac{5}{31}$ $\left(\text{the reciprocal of } -\frac{31}{5}\right)$ to get 1 in the third row and the third column position.

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 9 \\ 0 & -5 & 8 & -26 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Multiply each member of the second row by $-\frac{1}{5}$.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{8}{5} & \frac{26}{5} \\ 0 & 0 & 1 & -2 \end{array}\right]$$

We now have the system

$$\begin{aligned} x + 2y - 3z &= 9 \\ y - \frac{8}{5}z &= \frac{26}{5} \\ z &= -2 \end{aligned}$$

Replace z by -2 in $y - \frac{8}{5}z = \frac{26}{5}$ and solve for y.

$$\begin{aligned} y - \frac{8}{5}(-2) &= \frac{26}{5} \\ y + \frac{16}{5} &= \frac{26}{5} \\ y &= \frac{26}{5} - \frac{16}{5} = \frac{10}{5} = 2 \end{aligned}$$

Replace y by 2 and z by -2 in the equation $x + 2y - 3z = 9$.

$$\begin{aligned} x + 2(2) - 3(-2) &= 9 \\ x + 4 + 6 &= 9 \\ x + 10 &= 9 \\ x &= -1 \end{aligned}$$

The solution set of the system is $\{(-1,2,-2)\}$. ■

When systems of linear equations are inconsistent (no solution) or dependent (infinitely many solutions), the augmented matrix yields (1) a false statement for inconsistent or (2) a statement that is always true for dependent.

■ Example 8–6 C

Solve the following systems of equations using an augmented matrix.

1. $3x - 2y = 6$
   $6x - 4y = 1$

$$\left[\begin{array}{cc|c} 3 & -2 & 6 \\ 6 & -4 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 2 \\ 6 & -4 & 1 \end{array}\right] \quad \text{Multiply row 1 by } \frac{1}{3}$$

$$= \left[\begin{array}{cc|c} 1 & -\frac{2}{3} & 2 \\ 0 & 0 & -11 \end{array}\right] \quad \text{Multiply row 1 by } -6 \text{ and add to row 2}$$

The second row yields $0 = -11$ (false) and the system is inconsistent.

2. $4x + 6y = -2$
$2x + 3y = -1$

$$\left[\begin{array}{cc|c} 4 & 6 & -2 \\ 2 & 3 & -1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -\frac{1}{2} \\ 2 & 3 & -1 \end{array}\right] \quad \text{Multiply row 1 by } \frac{1}{4}$$

$$= \left[\begin{array}{cc|c} 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \quad \text{Multiply row 1 by } -2 \text{ and add to row 2}$$

The second row yields $0 = 0$ (true) and the system is dependent.

▶ ***Quick check*** Solve the system of equations using an augmented matrix.
$4x - 5y = -33$
$-x + 3y = 17$

Mastery points

Can you

- Solve a system of two linear equations in two variables using the augmented matrix method?
- Solve a system of three linear equations in three variables using the augmented matrix method?

Exercise 8–6

Solve the following systems of linear equations using the augmented matrix method. See examples 8–6 A, B, and C.

Example $4x - 5y = -33$
$-x + 3y = 17$

Solution The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -5 & -33 \\ -1 & 3 & 17 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{4} & -\frac{33}{4} \\ -1 & 3 & 17 \end{array}\right] \quad \text{Multiply row 1 by } \frac{1}{4}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{4} & -\frac{33}{4} \\ 0 & \frac{7}{4} & \frac{35}{4} \end{array}\right] \quad \text{Add numbers in the first row to the numbers in the second row.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{4} & -\frac{33}{4} \\ 0 & 1 & 5 \end{array}\right] \quad \text{Multiply the second row by } \frac{4}{7}.$$

The augmented matrix yields the system

$$x - \frac{5}{4}y = -\frac{33}{4}$$
$$y = 5$$

Replace y with 5 in

$$x - \frac{5}{4}y = -\frac{33}{4}$$
$$x - \frac{5}{4}(5) = -\frac{33}{4}$$
$$x - \frac{25}{4} = -\frac{33}{4}$$
$$x = -\frac{8}{4} = -2$$

The solution set is $\{(-2,5)\}$.

1. $x + 3y = 11$
$2x - y = 1$

2. $x - 5y = 11$
$2x + 3y = -4$

$\boxed{\textbf{3.}}$ $x - 4y = -6$
$3x + y = -5$

4. $x + 6y = -14$
$5x - 3y = -4$

5. $2x + y = 5$
$3x - 5y = 14$

6. $3x - 2y = 16$
$4x + 2y = 12$

7. $5x - y = 0$
$2x + 3y = -1$

8. $-3x + 2y = 1$
$x - y = 4$

9. $-x - 2y = 4$
$x + 6y = -2$

$\boxed{\textbf{10.}}$ $-x - y = 4$
$2x + 2y = -1$

11. $4x - 2y = 1$
$-8x + 4y = -2$

12. $x + 3y = 1$
$3x + 9y = 3$

13. $-3x - y = 1$
$4x + 5y = -5$

$\boxed{\textbf{14.}}$ $x + 3y - z = 5$
$3x - y + 2z = 5$
$x + y + 2z = 7$

15. $x - 2y + 3z = -11$
$2x + 3y - z = 6$
$3x - y - z = 3$

16. $2x + 2y - z = 3$
$x - 4y + 5z = 0$
$-x + y + z = 2$

17. $x - y + 3z = -1$
$2x + y - z = 0$
$-3x - 4y + z = 1$

18. $3x - y - 6z = 5$
$-6x + 2y - 2z = -4$
$3x - y + z = 2$

19. $2x - y + 3z = 5$
$3x - y + 2z = 10$
$3x - 2y + 7z = 3$

20. $x + 2z = 5$
$2x - y = 4$
$2y - z = 5$

21. $x - 2y + z = -2$
$3x + y = 7$
$2x - z = 0$

22. $2x - y + z = 8$
$x - 2y - 3z = 4$
$3x + 3y - z = -4$

23. $x - 2y - 2z = 4$
$2x + y - 3z = 7$
$x - y - z = 3$

24. $2x - y - z = -4$
$x + 3y - 4z = 12$
$x + y + z = -5$

25. $2x + 3y + z = 11$
$3x - y - z = 11$
$x - 2y - 5z = 2$

26. $2x - y = -1$
$2y - z = 6$
$x + z = 1$

27. $x - y = 1$
$2x - z = 0$
$2y - z = -2$

Translate each problem into a system of linear equations and solve the system using the augmented matrix method.

28. A plane travels 525 miles with the wind in $1\frac{1}{2}$ hours. The return trip against the wind takes $2\frac{1}{10}$ hours. Find the speed of the plane and the speed of the wind.

29. Find the values of a and b so that the points $(-2,1)$ and $(1,3)$ lie on the graph of $ax + by = 4$.

30. A total of \$5,000 is invested by Erin, part at 8% and part at 12%. How much did she invest at each rate if the return from both investments is the same?

31. Salesman Jim Connor must sell 24 new cars to meet his sales quota. He must sell an equal number of intermediate-sized and large-sized cars. If he must sell 3 more small-sized cars than intermediate-sized cars, how many of each size car must he sell?

Review exercises

Find the x- and y-intercepts of the following equations. See section 7–1.

1. $y = 2x + 8$
2. $3x - 2y = -9$
3. $y = 6$

Complete the square of the following to obtain a binomial square. See section 6–2.

4. $x^2 + 8x$
5. $y^2 - 5y$
6. $z^2 - \frac{1}{2}z$

Factor the following expressions. See section 3–5.

7. $y^2 - 14y + 49$
8. $x^2 + 10x + 25$

Chapter 8 lead-in problem

A total of 12,000 persons paid \$240,375 to attend a rock concert. If only two types of tickets were sold, one selling for \$17.50 and the other selling for \$25.00, how many of each type of tickets were sold?

Solution

Let x represent the number of \$17.50 tickets sold and y represent the number of \$25.00 tickets sold. We set up a table to help determine the equations.

	Number sold	Cost per ticket	Total income
First type	x	17.50	$17.50x$
Second type	y	25.00	$25.00y$
Totals	12,000		240,375

The system of linear equations is

$$\begin{aligned} x + y &= 12{,}000 \qquad (1) \\ 17.50x + 25.00y &= 240{,}375 \qquad (2) \end{aligned}$$

Multiply each member of (2) by 10 to clear decimal numbers. We then get

$$\begin{array}{rl} x + y = 12{,}000 & \xrightarrow{\text{Multiply by } -175} \; -175x - 175y = -2{,}100{,}000 \\ 175x + 250y = 2{,}403{,}750 & \qquad\qquad \underline{175x + 250y = \;\; 2{,}403{,}750} \\ & \qquad\qquad\qquad\quad 75y = \;\; 303{,}750 \quad \text{Add members} \\ & \qquad\qquad\qquad\qquad y = \;\; 4{,}050 \end{array}$$

Using equation (1),

$$\begin{aligned} x + y &= 12{,}000 \\ x + 4{,}050 &= 12{,}000 \quad \text{Replace } y \text{ with } 4{,}050 \\ x &= 7{,}950 \end{aligned}$$

Thus, 4,050 tickets were sold at \$25.00 and 7,950 were sold at \$17.50.

Chapter 8 summary

1. Two or more linear equations that involve the same variables are called a **system of linear equations.**
2. A system of equations is **consistent and independent** if the system has only one solution.
3. A system of equations is **dependent** if all solutions of one equation are also solutions of the other equation(s).
4. A system of equations is **inconsistent** if the system has no solution.
5. A system of linear equations can be solved by **elimination, substitution,** or by **determinants** using Cramer's Rule.
6. A **matrix** is an ordered array of rows and columns of numbers.
7. The 2 × 2 determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is defined by $a_1b_2 - a_2b_1$
8. By definition, the determinant
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$
9. By Cramer's Rule, given the system of linear equations
$a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$
$$x = \frac{D_x}{D} \text{ and } y = \frac{D_y}{D}$$
where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} (D \neq 0)$,
$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \text{ and } D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$
10. The augmented matrix of the system of linear equations $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ is given by
$$\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$$

Chapter 8 error analysis

1. Solving systems of linear equations by elimination
Example:
$$\begin{array}{ll} 3x - 2y = 4 & \qquad 3x - 2y = 4 \\ x + 4y = -1 \text{ Multiply by } 3 \rightarrow & \underline{3x + 12y = -3} \\ & 6x - 10y = -3 \text{ Add} \end{array}$$
Correct answer:
$$\begin{array}{ll} 3x - 2y = 4 & \qquad 3x - 2y = 4 \\ x + 4y = -1 \text{ Multiply by } -3 \rightarrow & \underline{-3x - 12y = 3} \\ & -14y = 7 \text{ Add} \\ & y = -\dfrac{1}{2} \end{array}$$
What error was made? (*see page 351*)
2. Check solutions of systems of equations
Example: (2,3) is a solution of the system
$4x - 3y = -1$
$2x + y = 6$
Check: $4(2) - 3(3) = -1$
$8 - 9 = -1$
$-1 = -1$ (True)
Correct answer: (2,3) not a solution.
What error was made? (*see page 349*)

3. Evaluating determinants

Example: $\begin{vmatrix} 2 & -3 \\ 4 & 1 \end{vmatrix} = (2)(1) + (4)(-3)$

$= 2 - 12 = -10$

Correct answer: 14

What error was made? (*see page 376*)

4. Evaluating 3 × 3 determinants

Example:

$$\begin{vmatrix} 2 & 0 & 1 \\ 3 & 2 & -2 \\ -1 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & -2 \\ -3 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ -3 & 0 \end{vmatrix}$$

$$-(-1)\begin{vmatrix} 0 & 1 \\ -3 & 0 \end{vmatrix} = 2(0-6) - 3(0+3) + 1(0+3)$$

$$= -12 - 9 + 3 = -18$$

Correct answer: -23

What error was made? (*see page 377*)

5. Solving systems by Cramer's Rule

Example: Given the system of linear equations

$3x - 2y = -5$

$2x + 5y = 4$

$$D_x = \begin{vmatrix} 3 & -2 \\ 2 & 5 \end{vmatrix} = 15 - (-4) = 19$$

Correct answer: $D_x = -17$

What error was made? (*see page 381*)

6. Solving systems by Cramer's Rule

Example: Given $2x - 3y + z = 1$

$x + 2y - z = -2$

$-3x - y + 4z = 0$

$$z = \frac{D_z}{D} = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ -3 & 0 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -3 & 1 \\ 1 & 2 & -1 \\ -3 & -1 & 4 \end{vmatrix}}$$

$$= \frac{2(-8) - 1(4) + (-3)(1)}{2(7) - 1(-11) + (-3)(1)} = \frac{-23}{22}$$

Correct answer: $\frac{-17}{22}$

What error was made? (*see page 383*)

7. Solving a system of equations by augmented matrix

Example: Given $x - y = 4$

$x + 3y = -1$

$$\left[\begin{array}{cc|c} 1 & -1 & 4 \\ 1 & 3 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 2 & -5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 4 \\ 0 & 1 & -\frac{5}{2} \end{array}\right]$$

$y = -\frac{5}{2}$

Correct answer: $y = -\frac{5}{4}$

What error was made? (*see page 388*)

8. Order of operations

Example:

$3^2 + 12 \div 3 - [5 - (3 + 6)]$

$= 9 + 12 \div 3 - 5 - 3 - 6$

$= 21 \div 3 - 14$

$= 7 - 14 = -7$

Correct answer: 17

What error was made? (*see page 27*)

9. Negative and rational exponents

Example: $(2m^{-2}n^{1/4})^{-1/4} = -\frac{1}{2}m^{1/2}$

Correct answer: $\frac{m^{1/2}}{2^{1/4}n^{1/16}}$

What error was made? (*see page 224*)

Chapter 8 critical thinking

Pick any integer from 1 to 99 and double it. Then add 8 and divide that sum by 2. If you subtract the original number from this result, your answer is always 4. Why is this true?

Chapter 8 review

[8–1]

Find the solution set of each system of linear equations by elimination. If the system is inconsistent or dependent, so state.

1. $x + y = 4$
$x - y = 2$

2. $2x + 3y = 2$
$2x - 3y = 0$

3. $2x + 3y = 4$
$4x + 6y = 8$

4. $x - 4y = 5$
$3x - 12y = 0$

5. $x + y = 1$
$4x - 4y = 6$

6. $\frac{1}{2}x + y = 4$
$x - \frac{1}{2}y = -5$

7. Forces F_1 and F_2 on a structure yield the system of equations

$$\frac{1}{3}F_1 + \frac{2}{3}F_2 = 3$$
$$\frac{2}{3}F_1 - \frac{1}{3}F_2 = 5$$

Find the forces F_1 and F_2.

Find the solution set of each system of linear equations by substitution. If the system is inconsistent or dependent, so state.

8. $3x + 2y = 1$
$y = -4$

9. $4x - y = 6$
$x = -1$

10. $5y - x = 1$
$y = 3x + 1$

11. $6x + 2y = -3$
$x = 4 - y$

12. $x - 4y = 1$
$2y - 2x = 3$

[8–2]

Set up a system of linear equations and solve each problem.

13. Find the point of intersection of the two lines L_1 and L_2 if L_1 contains the points $(-3,2)$ and $(1,0)$ and L_2 contains the points $(5,1)$ and $(-2,-3)$.

14. Noel Doe wishes to enclose her rectangular yard with 180 feet of fencing. If she wishes to make the enclosed yard twice as long as it is wide, what are the dimensions of the yard? (*Hint:* The perimeter $P = 2\ell + 2w$. Set up two equations in ℓ and w and solve simultaneously.)

15. The total number of fire alarms in Detroit on a given day is 30. If the number of real alarms is two more than six times the number of false alarms, how many of each alarm are sounded?

16. A woman has \$3,000 to invest. If she invests part at 7% simple interest and the rest at $6\frac{1}{2}\%$ simple interest, and the total income for the year is \$201, how much does she invest at each rate?

17. Solder made of 20% tin is to be melted with solder made of 5% tin to produce 50 grams of solder containing 10% tin. How many grams of each should be used?

18. George bought a suit and a topcoat for \$177. If one-fifth of the cost of the suit is \$9 more than one-sixth of the cost of the topcoat, what is the price of each article of clothing?

19. Two airplanes leave from Detroit, one flying East and the other flying West. If one plane flies at 250 mph and the other flies at 305 mph, in how many hours will they be 2,886 miles apart?

[8–3]

Find the solution set of each system of three linear equations. If the system is inconsistent or dependent, so state.

20. $x - y + 2z = 5$
$x + y + z = 6$
$2x - y - z = -3$

21. $7x - 2y + 9z = -3$
$4x - 6y + 8z = -5$
$8x + y + z = 1$

22. $2u + v + 3w = -2$
$5u + 2v = 5$
$2v - 3w = -7$

23. $p - r = -1$
$-q + r = 2$
$-2p + q = -4$

24. Find the values of a, b, and c such that the points (0,2), (1,−3), and (2,−2) lie on the graph of $y = ax^2 + bx + c$.

25. Three forces on a beam are related by the system of equations
$0.2F_1 + 0.3F_2 = 2$
$0.2F_1 - 0.1F_3 = 1$
$0.4F_2 + 0.2F_3 = 3$
Find forces F_1, F_2, and F_3.

26. If the middle-sized angle of a triangle measures 16° more than the smallest angle and the largest angle is twice the size of the smallest angle, find the measure of the three angles of the triangle.

27. Ron has a total of 285 coins consisting of nickels, dimes, and quarters in his piggy bank. If there is a total of \$26.25 in the bank and there are four times as many dimes and quarters, how many of each coin does he have in the bank?

[8–4]

Evaluate each determinant.

28. $\begin{vmatrix} 1 & 4 \\ -1 & 3 \end{vmatrix}$

29. $\begin{vmatrix} 0 & -6 \\ 4 & 3 \end{vmatrix}$

30. $\begin{vmatrix} 5 & 7 \\ -8 & -3 \end{vmatrix}$

Evaluate each determinant using expansion about any row or column.

31. $\begin{vmatrix} 3 & 4 & 3 \\ -2 & 2 & 0 \\ 1 & -5 & 6 \end{vmatrix}$

32. $\begin{vmatrix} -3 & 3 & 2 \\ -1 & 0 & -4 \\ -2 & 0 & 5 \end{vmatrix}$

33. $\begin{vmatrix} 0 & 1 & -3 \\ -2 & 0 & 5 \\ 6 & 7 & 8 \end{vmatrix}$

[8–5]

Use Cramer's Rule to find the solution set of each system of linear equations.

34. $2x - y = 3$
$x - 3y = 4$

35. $4x + 5y = 0$
$2x - 4y = -1$

36. $-4x + 2y = 3$
$y = 9$

37. $6x - 3y = -2$
$3x = 8$

38. $x - 3y + 2z = 0$
$2x - y + z = 0$
$x + 4y - 3z = 0$

39. $-4x + y - 3z = -2$
$3x - 2y + z = 4$
$x + 3y - 2z = 1$

[8–6]

Use the augmented matrix method to solve the following systems of linear equations.

40. $3x - y = 2$
$x + 2y = 0$

41. $x - y + 2z = 3$
$x + 3y - z = 1$
$2x - y + 2z = 0$

Chapter 8 cumulative test

[1–4] 1. Simplify $3[-4(12 - 3) - 14 + 6(3 - 9)]$.

Evaluate the following formulas.

[1–5] 2. $V = k + gt$ when $k = 14$, $g = 32$, and $t = 3$

[1–5] 3. $C = \dfrac{C_1C_2}{C_1 + C_2}$ when $C_1 = 8$ and $C_2 = 12$

Perform the indicated operations and simplify.

[4–5] 4. $\dfrac{3a^3 - 11a^2 + 12a - 3}{a - 3}$

[4–5] 5. $\dfrac{30a^7 - 25a^5 + 15a^3}{5a^2}$

[3–3] 6. $(5x + 1)(x - 9) - (x + 6)^2$

[3–3] 7. $\dfrac{3^{-1}x^0y^{-2}}{6^{-1}x^{-4}y^3}$

Find the solution set of the following equations and inequalities (problems 8–16).

[2–1] 8. $5(6y - 1) - 3(4y + 3) = 5y$

[6–3] 9. $3x^2 - 2x - 5 = 0$ (Use the quadratic formula.)

[6–3] 10. $x^2 + 6x - 5 = 0$ (Use the quadratic formula.)

[6–2] 11. $2^7y^2 + y = 5$ (Use completing the square.)

[2–5] 12. $3(z + 1) - 1 \le 2(3z + 3)$

[6–7] 13. $x^2 + 2x - 3 \le 0$

[6–7] 14. $2y^2 + 9y > 5$

[2–4] 15. $|2x - 1| = 4$

[2–6] 16. $|5 - 6x| > 2$

[4–1] 17. Reduce the expression $\dfrac{8y + 12}{4y^2 - 9}$ to lowest terms.

Perform the indicated operations and simplify.

[4–2] 18. $\dfrac{x^2 - 16}{4x - 3} \cdot \dfrac{16x^2 - 9}{3x + 12}$

[4–3] 19. $\dfrac{11}{x^2 - x - 20} - \dfrac{7}{x^2 + x - 12}$

[4–3] 20. $\dfrac{3y}{y - 7} + \dfrac{y}{7 - y}$

[4–2] 21. $\dfrac{p^3q}{16ab^2} \div \dfrac{7pq^2}{24a^2b^3}$

Leave all answers with positive exponents. Assume all variables are positive.

[5–2] 22. $(-27)^{2/3}$

[5–2] 23. $(y^{3/2})^{-2}$

[5–2] 24. $\dfrac{a^{3/4}}{a^{-1/2}}$

[5–4] 25. $\sqrt{8} \cdot \sqrt{10}$

[5–6] 26. $(\sqrt{6} + 3)(\sqrt{2} - 4)$

[5–7] 27. $(4 + 5i)^2$

[5–5] 28. $\sqrt[3]{8xy^3} - \sqrt[3]{64xy^3}$

[6–5] 29. Find the solution set of the equation $p + 7\sqrt{p} + 6 = 0$. Identify any extraneous solutions that exist.

[7–1] 30. Graph the equation $4x + 5y = -20$ using the intercepts.

[7–3] 31. Graph the equation $3x - y = 6$ using the slope and y-intercept.

[7–3] 32. Find the equation of the line through the points $(0,4)$ and $(-7,9)$.

[7–3] 33. Find the equation of the line through $(1,3)$ and perpendicular to the line $4x - y = 5$.

[7–4] 34. Sketch the graph of the inequality $2y + x < 2$.

[8–4] 35. Evaluate $\begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & 1 \\ 5 & -1 & 4 \end{vmatrix}$

[8–1] 36. $3x + y = 6$
$x - y = 2$

[8–1] 37. $2y + 3x = 1$
$y - x = -4$

[8–5] 38. $4x + 3y = 0$
$3x - 2y = 1$
(By determinants)

[8–6] 39. $-3x - y = 2$
$4x + 5y = -5$
(By augmented matrix)

[8–3] 40. $x + 4y - z = -3$
$-2x + y + 2z = 0$
$3x - 2y + z = 1$

14. $\{y \mid y \le -4 \text{ or } y \ge -2\} = (-\infty, -4] \cup [-2, \infty)$

15. $\left\{0, \frac{1}{4}\right\}$ **16.** $\{9, -2\}$ **17.** $\left\{\frac{3}{2}, -1\right\}$

18. $\left\{x \,\middle|\, -\frac{1}{3} \le x < 3\right\} = \left[-\frac{1}{3}, 3\right)$ **19.** $\left\{\frac{3+\sqrt{65}}{4}, \frac{3-\sqrt{65}}{4}\right\}$

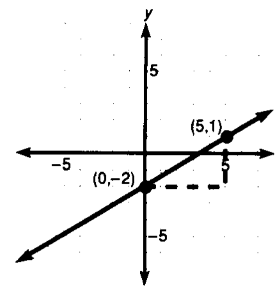
20. $\frac{1}{y-5}$ **21.** $\frac{-3y^2 + 46y - 21}{2y(y+7)(y-7)}$ **22.** 1 **23.** $\left\{\frac{11}{10}\right\}$

24. **a.** $P(-3) = 0$ **b.** $x + 3$ is a factor of $3x^3 + 8x^2 - 7x = 12$

25. $6\sqrt{2}$ **26.** $21 + 8\sqrt{5}$ **27.** $\frac{3\sqrt{5}}{5}$ **28.** $6 + 3\sqrt{3}$ **29.** 13

30. $2 - 7i$ **31.** $\emptyset$; 7 is extraneous **32.** $7x - y = 10$

33. $2x - 3y = -8$ **34.** $x = 5$ **35.** $m = \frac{3}{5}, b = -2$



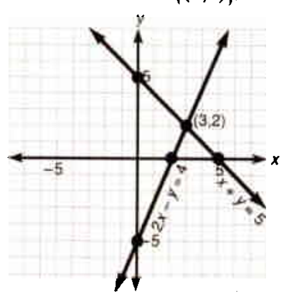
36. perpendicular **37.** $d = \sqrt{65}$; midpoint, $\left(\frac{3}{2}, 1\right)$

Chapter 8

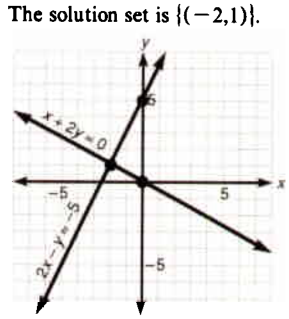
Exercise 8–1

Answers to odd-numbered problems

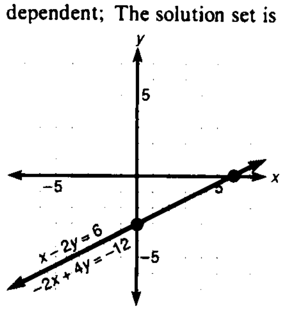
1. yes **3.** yes **5.** $(-1,2)$ not a solution **7.** The solution set is $\{(3,2)\}$.



9. The solution set is $\{(-2,1)\}$.



11. dependent; The solution set is $\{(x,y) \mid x - 2y = 6\}$. **13.** $\{(-2,-5)\}$



15. $\{(-2,3)\}$ **17.** $\{(2,3)\}$ **19.** $\{(3,1)\}$ **21.** $\left\{\left(\frac{3}{2}, 1\right)\right\}$

23. $\{(-1,-4)\}$ **25.** $\{(x,y) \mid 3x + y = 2\}$; dependent

27. $\emptyset$; inconsistent **29.** $\left\{\left(\frac{5}{9}, \frac{10}{9}\right)\right\}$ **31.** $\left\{\left(\frac{5}{3}, \frac{1}{2}\right)\right\}$

33. $\left\{\left(\frac{3}{5}, 0\right)\right\}$ **35.** $\left\{\left(-2, \frac{7}{2}\right)\right\}$ **37.** $\{(3,-2)\}$

39. $\{(7,-4)\}$ **41.** $\left\{\left(\frac{7}{2}, -\frac{3}{2}\right)\right\}$ **43.** $\{(1,-3)\}$ **45.** $\{(-1,-4)\}$

47. $\{(4,12)\}$ **49.** $\{(-2,-5)\}$ **51.** $\{(6,6)\}$ **53.** $\{(-1,0)\}$

55. $\emptyset$; inconsistent **57.** $\{(x,y) \mid 2x - y = 7\}$; dependent

59. $\left\{\left(\frac{5}{2}, 2\right)\right\}$ **61.** $\left\{\left(\frac{3}{8}, \frac{33}{8}\right)\right\}$ **63.** $\left\{\left(-\frac{12}{11}, -\frac{63}{11}\right)\right\}$

65. $\left\{\left(-1, \frac{1}{4}\right)\right\}$ **67.** $\left\{\frac{7}{5}, -\frac{7}{4}\right\}$ **69.** $x + y = 502$

71. $x = y + 6$ or $y = x + 6$ **73.** $y = 3x + 4$ or $x = 3y + 4$

75. $x - y = 33$

Solutions to trial exercise problems

19. $\begin{aligned} 3x + 2y &= 11 \\ \underline{x - y} &\underline{= 2} \quad \text{(times 2)} \end{aligned}$ $\quad \begin{aligned} 3x + 2y &= 11 \\ \underline{2x - 2y} &\underline{= 4} \\ 5x \quad &= 15 \\ x &= 3 \end{aligned}$ $\quad \begin{aligned} 3 - y &= 2 \\ -y &= -1 \\ y &= 1 \end{aligned}$ The solution set is $\{(3,1)\}$.

24. $\begin{aligned} 3x - y &= 10 \quad \text{(times } -2) \\ \underline{6x - 2y} &\underline{= 5} \end{aligned}$ $\quad \begin{aligned} -6x + 2y &= -20 \\ \underline{6x - 2y} &\underline{= 5} \\ 0 &= -15 \end{aligned}$ The system is inconsistent. There are no common solutions. The solution set is $\emptyset$.

31. $\begin{aligned} \frac{1}{2}x + \frac{1}{3}y &= 1 \quad \text{(times 6)} \\ \frac{1}{4}x - \frac{2}{3}y &= \frac{1}{12} \quad \text{(times 12)} \end{aligned}$ $\quad \begin{aligned} 3x + 2y &= 6 \\ \underline{3x - 8y} &\underline{= 1} \quad \text{(times } -1) \end{aligned}$ $\quad \begin{aligned} 3x + 2y &= 6 \\ \underline{-3x + 8y} &\underline{= -1} \\ 10y &= 5 \\ y &= \frac{1}{2} \end{aligned}$ $\quad \begin{aligned} \frac{1}{2}x + \frac{1}{3}\left(\frac{1}{2}\right) &= 1 \\ \frac{1}{2}x + \frac{1}{6} &= 1 \\ 3x + 1 &= 6 \\ 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$

The solution set is $\left\{\left(\frac{5}{3}, \frac{1}{2}\right)\right\}$.

36. $\begin{aligned} (0.3)x - (0.8)y &= 1.6 \quad \text{(times 10)} \\ \underline{(0.1)x + (0.4)y} &\underline{= 1.2} \quad \text{(times 10)} \end{aligned}$ $\quad \begin{aligned} 3x - 8y &= 16 \\ \underline{x + 4y} &\underline{= 12} \quad \text{(times 2)} \end{aligned}$ $\quad \begin{aligned} 3x - 8y &= 16 \\ \underline{2x + 8y} &\underline{= 24} \\ 5x \quad &= 40 \\ x &= 8 \end{aligned}$ $\quad \begin{aligned} 8 + 4y &= 12 \\ 4y &= 12 - 8 = 4 \\ y &= 1 \end{aligned}$

The solution set is $\{(8,1)\}$.

39. $\begin{aligned} 2x + y &= 10 \\ \underline{y} &\underline{= -x + 3} \end{aligned}$ $\quad \begin{aligned} 2x + (-x + 3) &= 10 \\ x + 3 &= 10 \\ x &= 7 \end{aligned}$ $\quad \begin{aligned} y &= -(7) + 3 \\ y &= -4 \end{aligned}$ The solution set is $\{(7,-4)\}$.

44. $\begin{aligned} 3x - 5y &= 4 \\ \underline{x + 2y} &\underline{= -2} \end{aligned}$ $\quad x = -2y - 2,$

substituting $\begin{aligned} 3(-2y - 2) - 5y &= 4 \\ -6y - 6 - 5y &= 4 \\ -11y &= 10 \\ y &= -\frac{10}{11} \end{aligned}$

Then $x = -2\left(-\frac{10}{11}\right) - 2 = \frac{20}{11} - 2 = -\frac{2}{11}$

The solution set is $\left\{\left(-\frac{2}{11}, -\frac{10}{11}\right)\right\}$.

49. $\begin{aligned} 4x - 3y &= 7 \\ \underline{y} &\underline{= -5} \end{aligned}$ Substituting,

$$\begin{aligned} 4x - 3(-5) &= 7 \\ 4x + 15 &= 7 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

The solution set is $\{(-2,-5)\}$.

61. $-\frac{1}{3}x + y = 4 \quad (3) \quad -x + 3y = 12$

$x = \frac{1}{3}y - 1 \quad (3) \quad 3x = y - 3,$ then $y = 3x + 3$.

Substituting in $-x + 3y = 12$, $\begin{aligned} -x + 3(3x + 3) &= 12 \\ -x + 9x + 9 &= 12 \\ 8x &= 3 \\ x &= \frac{3}{8} \end{aligned}$

Then, $y = 3\left(\frac{3}{8}\right) + 3 = \frac{9}{8} + 3 = \frac{33}{8}$.

The solution set is $\left\{\left(\frac{3}{8}, \frac{33}{8}\right)\right\}$.

66. Let $p = \frac{1}{x}$ and $q = \frac{1}{y}$.

$$\frac{2}{x} - \frac{3}{y} = 1 \qquad 2p - 3q = 1$$
$$\frac{1}{x} + \frac{2}{y} = 2 \qquad p + 2q = 2 \qquad \text{Multiply by } -2$$

$$\begin{aligned} 2p - 3q &= 1 \\ \underline{-2p - 4q} &\underline{= -4} \\ -7q &= -3 \\ q &= \frac{3}{7} \\ p + 2\left(\frac{3}{7}\right) &= 2 \\ p + \frac{6}{7} &= 2 \\ p &= \frac{8}{7} \end{aligned}$$

$$\frac{1}{x} = \frac{8}{7} \text{ implies } x = \frac{7}{8}$$
$$\frac{1}{y} = \frac{3}{7} \text{ implies } y = \frac{7}{3}$$

The solution set is $\left\{\left(\frac{7}{8}, \frac{7}{3}\right)\right\}$.

Review exercises

1. $-x^3 - 5x^2 - x + 5$ **2.** $x^3 + 8$ **3.** $25y^2 - 20yz + 4z^2$
4. $\{1\}$; -9 is extraneous **5.** 36 and 9 **6.** 12

Exercise 8-2

Answers to odd-numbered problems

1. 8 ft by 12 ft **3.** 9 m by 6 m **5.** 6 ft and 15 ft **7.** 11 volts and 36 volts **9.** \$6,000 at 6½%; \$14,000 at 7% **11.** \$15,000 at 6%; \$15,000 at 8% **13.** \$16,800 at 8%; \$19,200 at 7% **15.** \$9,000 at 12%; \$16,000 at 15% **17.** 15 suits at \$205; 17 suits at \$152 **19.** 23 laborers (5 cat operators) **21.** 403 children's tickets were sold (100 adult tickets were sold). **23.** 19 quarters; 24 dimes **25.** 1,600 kg of 85% pure copper; 400 kg of 60% pure copper **27.** 320 liters of 6% acid; 480 liters of 3.5% acid **29.** $28\frac{4}{7}$ ml **31.** 2 liters of pure antifreeze; 10 liters of 4% solution **33.** speed of boat is 20 mph; speed of stream is 4 mph **35.** The jogger runs for $\frac{6}{7}$ of an hour or approximately 51 min 26 sec. **37.** The mother jogs at 2 mph; the daughter jogs at 4 mph. **39.** cyclist's rate $= 12\frac{8}{9}$ mph; pedestrian's rate $= 4\frac{8}{9}$ mph **41.** 25 mph and 75 mph **43.** $x - 2y = -10$; $2x + y = 3$; $\left(-\frac{4}{5}, \frac{23}{5}\right)$ **45.** $\left(\frac{11}{5}, -\frac{8}{5}\right)$

Solutions to trial exercise problems

1. Let w = width of the rectangle. Let ℓ = length of the rectangle.

(1) Then $2\ell + 3w = 48$
$\underline{2\ell + 2w = 40}$ (times -1)

(2) $$\begin{aligned} 2\ell + 3w &= 48 \\ \underline{-2\ell - 2w} &\underline{= -40} \\ w &= 8 \end{aligned}$$

(3) $$\begin{aligned} 2\ell + 2(8) &= 40 \\ 2\ell + 16 &= 40 \\ 2\ell &= 24 \\ \ell &= 12 \end{aligned}$$

The room is 8 feet wide and 12 feet long.

12. Let x = amount invested at 7%. Let y = amount invested at 9%.
Then $x + y = 16{,}000$
$0.07x = 0.09y$

(1) $x + y = 16{,}000$ (times 9)
$7x - 9y = 0$

(2) $$\begin{aligned} 9x + 9y &= 144{,}000 \\ \underline{7x - 9y} &\underline{= 0} \\ 16x &= 144{,}000 \\ x &= 9{,}000 \end{aligned}$$
Then $y = 7{,}000$

Jamie invested \$9,000 at 7% and \$7,000 at 9%.

17. Let x = number of suits sold at \$152 each. Let y = number of suits sold at \$205 each.
Then $x + y = 32$ (times -152)
$152x + 205y = 5{,}659$

(1) $$\begin{aligned} -152x - 152y &= -4{,}864 \\ \underline{152x + 205y} &\underline{= 5{,}659} \\ 53y &= 795 \\ y &= 15 \end{aligned}$$

(2) $$\begin{aligned} x + 15 &= 32 \\ x &= 17 \end{aligned}$$

15 suits were sold at \$205 each and 17 were sold at \$152 each.

21. Let x = the number of children's tickets sold. Let y = the number of adult tickets sold.
$x + y = 503$
$1.25x + 3.50y = 853.75$

(1) $x + y = 503$ (times -125)
$125x + 350y = 85{,}375$

$$\begin{aligned} -125x - 125y &= -62{,}875 \\ \underline{125x + 350y} &\underline{= 85{,}375} \\ 225y &= 22{,}500 \\ y &= 100 \end{aligned}$$

(2) Then $x + 100 = 503$
$x = 403$

There were 403 children's tickets sold.

36. Let x = number of hours at 4.5 mph. Let y = number of hours at 4 mph.

Then $x + y = 7$
$4.5x + 4y = 30$

(1) $-4x - 4y = -28$
$\underline{4.5x + 4y = 30}$
$0.5x = 2$
$x = 4$

(2) $4 + y = 7$
$y = 3$

They rowed 4 hours at 4.5 mph and 3 hours at 4 mph.

39. Let x = rate of the cyclist. Let y = rate of the pedestrian.

Then $2\frac{1}{4}x + 2\frac{1}{4}y = 40$

$5x = 5y + 40$ or $x = y + 8$

(1) $\frac{9}{4}x + \frac{9}{4}y = 40$ (times 4)
$\underline{x - y = 8}$ (times 9)

(2) $9x + 9y = 160$
$\underline{9x - 9y = 72}$
$18x = 232$
$x = \frac{232}{18} = \frac{116}{9}$

(3) $\left(\frac{116}{9}\right) = y + 8$

$\frac{116}{9} = y + 8$

$y = \frac{116}{9} - 8 = \frac{44}{9}$

$y = \frac{44}{9}$

The cyclist travels at $\frac{116}{9} = 12\frac{8}{9}$ mph and the pedestrian walks at $\frac{44}{9} = 4\frac{8}{9}$ mph.

42. (1) Using $(-1,-2)$ and $(3,4)$, $m = \frac{4-(-2)}{3-(-1)} = \frac{6}{4} = \frac{3}{2}$.

Then $y - 4 = \frac{3}{2}(x - 3)$
$2y - 8 = 3x - 9$
$-3x + 2y = -1$

(2) Using $(4,1)$ and $(2,-4)$, $m = \frac{1-(-4)}{4-2} = \frac{5}{2}$.

Then $y - 1 = \frac{5}{2}(x - 4)$
$2y - 2 = 5x - 20$
$-5x + 2y = -18$

(3) Solving $-3x + 2y = -1$
$\underline{-5x + 2y = -18}$ (-1)

$-3x + 2y = -1$
$\underline{5x - 2y = 18}$
$2x = 17$

$x = \frac{17}{2}$

Then $-3\left(\frac{17}{2}\right) + 2y = -1$

$-\frac{51}{2} + 2y = -1$

$2y = \frac{49}{2}$

$y = \frac{49}{4}$

$\left(\frac{17}{2}, \frac{49}{4}\right)$

Review exercises

1. $2x - 3y = -7$ **2.** $x - 2y = 6$ **3.** $2x + y = 3$
4. -8 **5.** -3 **6.** $2i$ **7.** $2xy\sqrt{2y}$

Exercise 8–3

Answers to odd-numbered problems

1. $\{(3,1,2)\}$ **3.** $\{(3,1,2)\}$ **5.** $\{(2,3,1)\}$ **7.** $\{(3,-1,2)\}$
9. $\{(5,-5,7)\}$ **11.** inconsistent
13. dependent; $\{(x,y,z) | x - 4y + z = -5\}$ **15.** $\{(1,-2,1)\}$
17. $\left\{\left(\frac{43}{3}, -3, \frac{122}{3}\right)\right\}$ **19.** $\{(8,1,-5)\}$ **21.** $\{(-1,-3,2)\}$
23. $\left\{\left(\frac{2}{5}, -\frac{23}{5}, -2\right)\right\}$ **25.** $\{(1,0,3)\}$ **27.** $\left\{\left(8, \frac{5}{2}, \frac{5}{2}\right)\right\}$
29. dependent; $\{(x,y,z) | 2x + 8y - 2z = 6\}$
31. 33°, 47°, 100° **33.** 25 m, 36 m, 61 m **35.** p_1 (expensive) = \$21.00; p_2(middle-priced) = \$15.00; p_3(cheapest) = \$11.00
37. 16—\$750 stamps; 26—\$1,500 stamps; 4—\$25,000 stamps
39. 10—\$5 bills, 15—\$10 bills, 8—\$20 bills
41. $a = \frac{1}{2}, b = -\frac{9}{2}, c = 2$

Solutions to trial exercise problems

1. $x + y + z = 6$
$x - 2y - z = -1$
$\underline{x + y - z = 2}$

(1) Using $x + y + z = 6$
$\underline{x - 2y - z = -1}$
$2x - y = 5$

(2) Using $x + y + z = 6$
$\underline{x + y - z = 2}$
$2x + 2y = 8$

(3) Solving $2x - y = 5$ (times 2)
$\underline{2x + 2y = 8}$

(4) Then $4x - 2y = 10$
$\underline{2x + 2y = 8}$
$6x = 18$
$x = 3$

(5) Substituting
$2(3) - y = 5$
$6 - y = 5$
$-y = -1$
$y = 1$

(6) Substituting in $x + y + z = 6$
$3 + 1 + z = 6$
$4 + z = 6$
$z = 2$

The solution set is $\{(3,1,2)\}$.

8.
$$\begin{aligned} x + 2y + 3z &= 5 \\ -x + y - z &= -6 \\ 2x + y + 4z &= 4 \end{aligned}$$

(1) Using
$$\begin{aligned} x + 2y + 3z &= 5 \\ -x + y - z &= -6 \\ \hline 3y + 2z &= -1 \end{aligned}$$

(2) Using
$$\begin{aligned} -x + y - z &= -6 \\ 2x + y + 4z &= 4 \end{aligned} \qquad \text{(times 2)} \quad \begin{aligned} -2x + 2y - 2z &= -12 \\ 2x + y + 4z &= 4 \\ \hline 3y + 2z &= -8 \end{aligned}$$

(3) Solving
$$\begin{aligned} 3y + 2z &= -1 \\ 3y + 2z &= -8 \end{aligned} \qquad \text{(times } -1\text{)} \quad \begin{aligned} -3y - 2z &= 1 \\ 3y + 2z &= -8 \\ \hline 0 &= -7 \end{aligned}$$

The system is *inconsistent*. There are no common solutions. The solution set is $\emptyset$.

13.
$$\begin{aligned} x - 4y + z &= -5 \\ 3x - 12y + 3z &= -15 \\ -2x + 8y - 2z &= 10 \end{aligned}$$

Using
$$\begin{aligned} x - 4y + z &= -5 \quad \text{(times } -3\text{)} \\ 3x - 12y + 3z &= -15 \end{aligned} \qquad \begin{aligned} -3x + 12y - 3z &= 15 \\ 3x - 12y + 3z &= -15 \\ \hline 0 &= 0 \end{aligned}$$

The system is *dependent*. The solution set is $\{(x,y,z) | x - 4y + z = -5\}$ (or either of the other equations).

18.
$$\begin{aligned} x - y &= -1 \\ x + z &= -2 \\ y - z &= 2 \end{aligned}$$

(1) Using
$$\begin{aligned} x + z &= -2 \\ y - z &= 2 \\ \hline x + y &= 0 \end{aligned}$$

(2) Using
$$\begin{aligned} x - y &= -1 \\ x + y &= 0 \\ \hline 2x \quad &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

(3) Using $x - y = -1$ and substituting $-\frac{1}{2}$ for x
$$\begin{aligned} -\frac{1}{2} - y &= -1 \\ -y &= -\frac{1}{2} \\ y &= \frac{1}{2} \end{aligned}$$

(4) Using $y - z = 2$, $\frac{1}{2} - z = 2$,
$$\begin{aligned} -z &= \frac{3}{2} \\ z &= -\frac{3}{2} \end{aligned}$$

The solution set is $\left\{\left(-\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}\right)\right\}$.

33. Let $x =$ the length of the shortest side
$y =$ the length of the middle side
$z =$ the length of the longest side.

Then
$$\begin{aligned} x + y + z &= 122 \\ z &= x + y \\ 2x + 11 &= z \end{aligned} \quad \text{and} \quad \begin{aligned} x + y + z &= 122 \\ -x - y + z &= 0 \\ 2x - z &= -11 \end{aligned}$$

(1) Add
$$\begin{array}{r} x + y + z = 122 \\ \underline{-x - y + z = 0} \\ 2z = 122 \\ z = 61 \end{array}$$

(2) Substitute in $2x - z = -11$
$$\begin{aligned} 2x - 61 &= -11 \\ 2x &= 50 \\ x &= 25 \end{aligned}$$

(3) Substitute in $x + y + z = 122$
$$\begin{aligned} 25 + y + 61 &= 122 \\ y + 86 &= 122 \\ y &= 36 \end{aligned}$$

The sides have length 25 meters, 36 meters, and 61 meters.

40. (1) Using (0,5),
$5 = a(0)^2 + b(0) + c$
$c = 5$

(2) Using $(-1,2)$,
$2 = a - b + c$

(3) Using (2, 17),
$17 = a(2)^2 + b(2) + c$
$17 = 4a + 2b + c$

(4) Solve the system.
$$\begin{aligned} a - b + c &= 2 \\ 4a + 2b + c &= 17 \\ c &= 5 \end{aligned}$$

(5) Substitute 5 for c.
$$\begin{array}{r} a - b + 5 = 2 \\ \underline{4a + 2b + 5 = 17} \end{array} \qquad \begin{aligned} a - b &= -3 \quad \text{(times 2)} \\ 4a + 2b &= 12 \end{aligned} \qquad \begin{array}{r} 2a - 2b = -6 \\ \underline{4a + 2b = 12} \\ 6a = 6 \\ a = 1 \end{array}$$

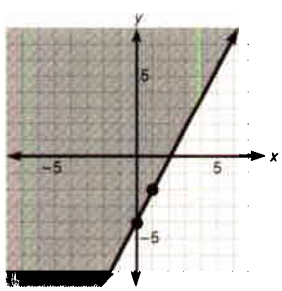
(6) Substitute 1 for a in
$$\begin{aligned} a - b &= -3 \\ 1 - b &= -3 \\ -b &= -4 \\ b &= 4 \end{aligned}$$

Therefore $a = 1$, $b = 4$, $c = 5$.

Review exercises

1. 2.47×10^{-4} **2.** $\dfrac{2y^2 - 6y + 18}{(y - 3)(y + 1)(y - 1)}$ **3.** 16

4. $2x - y \leq 4$ **5.** $\sqrt{65}, \left(-\dfrac{5}{2}, -1\right)$ **6.** $\left\{\dfrac{-2\sqrt{3}}{3}, \dfrac{2\sqrt{3}}{3}, -i, i\right\}$



Exercise 8-4

Answers to odd-numbered problems

1. 11 **3.** -24 **5.** 5 **7.** -35 **9.** -12 **11.** -14
13. 0 **15.** -4 **17.** 29 **19.** -14 **21.** -24 **23.** a^3
25. $y^3 - x^2y$ **27.** -19 **29.** 37

Solutions to trial exercise problems

3. $\begin{vmatrix} 4 & -2 \\ -6 & -3 \end{vmatrix} = -12 - (12) = -24$

9. $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}$ (Use first column.)
$= 1(6 - 1) - 3(6 - 3) + 2(2 - 6)$
$= 1(5) - 3(3) + 2(-4) = 5 - 9 - 8 = -12$

16. $\begin{vmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ 3 & -3 & 3 \end{vmatrix} = -2\begin{vmatrix} -1 & -1 \\ -3 & 3 \end{vmatrix} + 2\begin{vmatrix} -1 & -1 \\ 3 & 3 \end{vmatrix}$
$- 2\begin{vmatrix} -1 & -1 \\ 3 & -3 \end{vmatrix}$ (Use second row.)
$= -2(-3 - 3) + 2(-3 + 3) - 2(3 + 3)$
$= -2(-6) + 2(0) - 2(6) = 12 + 0 - 12 = 0$

23. $\begin{vmatrix} a & 0 & a \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} - 0\begin{vmatrix} 0 & a \\ 0 & a \end{vmatrix}$
$+ 0\begin{vmatrix} 0 & a \\ a & 0 \end{vmatrix}$ (Use first column.)
$= a(a^2 - 0) - 0(0 - 0) + 0(0 - a^2)$
$= a^3 - 0 + 0 = a^3$

27. $$\begin{vmatrix} 1 & 2 & 3 & -1 \\ 2 & 0 & 1 & 3 \\ -2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 0 \end{vmatrix} = 1\begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 3 & 2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 & 3 \\ -2 & 0 & -1 \\ 0 & 2 & 0 \end{vmatrix}$$

$$+ 3\begin{vmatrix} 2 & 0 & 3 \\ -2 & 1 & -1 \\ 0 & 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & 3 & 2 \end{vmatrix} \text{(Use first row.)}$$

$$= 1\left[0\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}\right] \text{(Use first column.)}$$

$$- 2\left[2\begin{vmatrix} 0 & -1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix}\right]$$

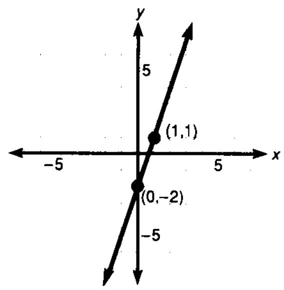
$$+ 3\left[2\begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} + 2\begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix}\right]$$

$$+ 1\left[2\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right]$$

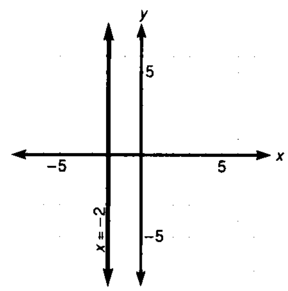
$$= 1[0 - 1(-6) + 3(-1)] - 2[2(2) + 2(-6) + 0]$$
$$+ 3[2(3) + 2(-9) + 0] + 1\,[2(2) + 2(-3) + 0]$$
$$= 1[6 - 3] - 2[4 - 12] + 3[6 - 18] + 1[4 - 6]$$
$$= 3 + 16 - 36 - 2 = 19 - 38 = -19$$

Review exercises

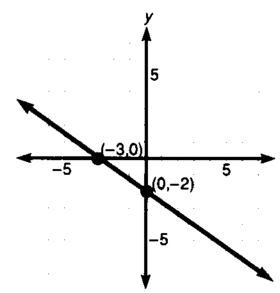
1. $\{(-2,2)\}$ **2.**



3.



4.



5. $P(-1) = 16$ **6.** $\{1,4\}$

Exercise 8–5

Answers to odd-numbered problems

1. $\left\{\left(\frac{5}{3},-\frac{1}{3}\right)\right\}$ **3.** $\left\{\left(\frac{-2}{19},\frac{24}{19}\right)\right\}$ **5.** $\emptyset$; inconsistent
7. $\{(x,y)|2x + 3y = -1\}$; dependent **9.** $\{(0,-1)\}$
11. $\left\{\left(\frac{5}{2},4\right)\right\}$ **13.** $\{(-2,5)\}$ **15.** $\{(9,-2)\}$ **17.** $\{(-1,2,3)\}$
19. $\{(0,0,0)\}$ **21.** $\{(x,y,z)|3x - y - 6z = 5\}$; dependent
23. $\emptyset$; inconsistent **25.** $\{(x,y,z)|3x + 2y + 4z = 3\}$; dependent **27.** $\emptyset$; inconsistent **29.** $\left\{\left(1,-6,-\frac{1}{2}\right)\right\}$
31. $\{(2,2,2)\}$ **33.** $\left\{\left(\frac{4}{3},3,\frac{8}{3}\right)\right\}$ **35.** $\{(-5,-2,1)\}$ **37.** 16 ft by 7 ft **39.** cream-filled—36¢; jelly-filled–32¢ **41.** 5—\$5 bills, 32—\$10 bills, 11—\$20 bills **43.** 3 elephants, 12 bears, 8 dogs

Solutions to trial exercise problems

1. $x - y = 2$
$2x + y = 3$

$$D = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 - (-2) = 3$$

$$D_x = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 - (-3) = 5$$

$$D_y = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1$$

Then $x = \dfrac{D_x}{D} = \dfrac{5}{3}$; $y = \dfrac{D_y}{D} = -\dfrac{1}{3}$

The solution set is $\left\{\left(\frac{5}{3},-\frac{1}{3}\right)\right\}$.

4. $4x - y = 3$
$8x - 2y = 1$

$$D = \begin{vmatrix} 4 & -1 \\ 8 & -2 \end{vmatrix} = -8 + 8 = 0$$

$$D_x = \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} = -6 + 1 = -5$$

$$D_y = \begin{vmatrix} 4 & 3 \\ 8 & 1 \end{vmatrix} = 4 - 24 = -20$$

Since $D = 0$ and at least one of D_x and $D_y \neq 0$, the system is *inconsistent*. The solution set is $\emptyset$.

11. $6x - 2y = 7$
$2y = 8$

$$D = \begin{vmatrix} 6 & -2 \\ 0 & 2 \end{vmatrix} = 12 - 0 = 12$$

$$D_x = \begin{vmatrix} 7 & -2 \\ 8 & 2 \end{vmatrix} = 14 - (-16) = 30$$

$$D_y = \begin{vmatrix} 6 & 7 \\ 0 & 8 \end{vmatrix} = 48 - 0 = 48$$

$$x = \frac{D_x}{D} = \frac{30}{12} = \frac{5}{2}; \quad y = \frac{D_y}{D} = \frac{48}{12} = 4$$

The solution set is $\left\{\left(\frac{5}{2}, 4\right)\right\}$.

17. $4x - y + 2z = 0$
$2x + y + z = 3$
$3x - y + z = -2$

$$D = \begin{vmatrix} 4 & -1 & 2 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= 4(1 + 1) - 2(-1 + 2) + 3(-1 - 2)$$
$$= 8 - 2 - 9 = -3$$

$$D_x = \begin{vmatrix} 0 & -1 & 2 \\ 3 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 0\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 2 \\ -1 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -1 & 2 \\ 1 & 1 \end{vmatrix}$$
$$= 0(1 + 1) - 3(-1 + 2) - 2(-1 - 2)$$
$$= 0 - 3 + 6 = 3$$

$$D_y = \begin{vmatrix} 4 & 0 & 2 \\ 2 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 2\begin{vmatrix} 0 & 2 \\ -2 & 1 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}$$
$$= 4(3 + 2) - 2(0 + 4) + 3(0 - 6)$$
$$= 20 - 8 - 18 = -6$$

$$D_z = \begin{vmatrix} 4 & -1 & 0 \\ 2 & 1 & 3 \\ 3 & -1 & -2 \end{vmatrix} = 4\begin{vmatrix} 1 & 3 \\ -1 & -2 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ -1 & -2 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix}$$
$$= 4(-2 + 3) - 2(2 + 0) + 3(-3 + 0)$$
$$= 4 - 4 - 9 = -9$$

$$x = \frac{D_x}{D} = \frac{3}{-3} = -1; \; y = \frac{D_y}{D} = \frac{-6}{-3} = 2; \; z = \frac{D_z}{D} = \frac{-9}{-3} = 3$$

The solution set is $\{(-1,2,3)\}$.

30. $x + y = 1$
$y + 2z = -2$
$2x - z = 0$

$$D = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= 1(-1 - 0) - 0(-1 - 0) + 2(2 - 0)$$
$$= 1(-1) - 0 + 2(2) = -1 + 4 = 3$$

$$D_x = \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$
$$= 1(-1 - 0) + 2(-1 - 0) + 0(2 - 0)$$
$$= 1(-1) + 2(-1) + 0 = -3$$

$$D_y = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 1\begin{vmatrix} -2 & 2 \\ 0 & -1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ -2 & 2 \end{vmatrix}$$
$$= 1(2 - 0) - 0(-1 - 0) + 2(2 - 0)$$
$$= 1(2) + 0 + 2(2) = 6$$

$$D_z = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & 0 & 0 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 0 & 0 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix}$$
$$= 1(0 - 0) - 0(0 - 0) + 2(-2 - 1)$$
$$= 0 - 0 + 2(-3) = -6$$

$$x = \frac{D_x}{D} = \frac{-3}{3} = -1; \; y = \frac{D_y}{D} = \frac{6}{3} = 2; \; z = \frac{D_z}{D} = \frac{-6}{3} = -2$$

The solution set is $\{(-1,2,-2)\}$.

37. Let w = width and ℓ = length of the rectangle.
Then $\ell = 2w + 2 \rightarrow \; -2w + \ell = 2$
$2\ell + 2w = 46 \rightarrow \; \underline{2w + 2\ell = 46}$

$$D = \begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} = -4 - 2 = -6$$

$$D_w = \begin{vmatrix} 2 & 1 \\ 46 & 2 \end{vmatrix} = 4 - 46 = -42$$

$$D_\ell = \begin{vmatrix} -2 & 2 \\ 2 & 46 \end{vmatrix} = -92 - 4 = -96$$

$$w = \frac{D_w}{D} = \frac{-42}{-6} = 7$$

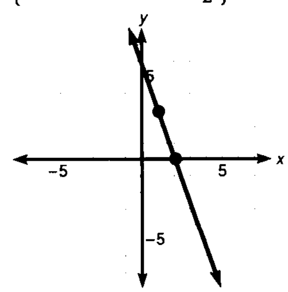
$$\ell = \frac{D_\ell}{D} = \frac{-96}{-6} = 16$$

The rectangle is 16 feet long and 7 feet wide.

Review exercises

1. $(-2,12)$; $(-1,9)$; $(0,6)$; $(1,3)$; $(2,0)$

2. $\left\{x \mid x < -3 \text{ or } x > \frac{1}{2}\right\} = (-\infty,-3) \cup \left(\frac{1}{2},\infty\right)$



3. $\left\{\frac{1}{3},5\right\}$ **4.** $\{-3,1\}$ **5.** $37 - 20\sqrt{3}$ **6.** 23 **7.** i

Exercise 8–6

Answers to odd-numbered problems

1. $\{(2,3)\}$ **3.** $\{(-2,1)\}$ **5.** $\{(3,-1)\}$ **7.** $\left\{\left(-\frac{1}{17},-\frac{5}{17}\right)\right\}$

9. $\left\{\left(-5,\frac{1}{2}\right)\right\}$ **11.** $\{(x,y) \mid 4x - 2y = 1\}$; dependent

13. $\{(0,-1)\}$ **15.** $\left\{\left(\frac{1}{5},\frac{4}{5},-\frac{16}{5}\right)\right\}$ **17.** $\left\{\left(-\frac{1}{19},-\frac{6}{19},-\frac{8}{19}\right)\right\}$

19. $\emptyset$; inconsistent **21.** $\left\{\left(\frac{4}{3},3,\frac{8}{3}\right)\right\}$ **23.** $\{(2,0,-1)\}$

25. $\{(4,1,0)\}$ **27.** dependent; $\{(x,y,z) \mid x - y = 1\}$ $\cap \{(x,y,z) \mid 2x - z = 0\} \cap \{(x,y,z) \mid 2y - z = -2\}$

29. $a = -\frac{8}{7}$, $b = \frac{12}{7}$ **31.** 10 small-sized, 7 intermediate-sized, 7 large-sized

Solutions to trial exercise problems

3. $x - 4y = -6$
$3x + y = -5$ augmented matrix $\left[\begin{array}{cc|c} 1 & -4 & -6 \\ 3 & 1 & -5 \end{array}\right]$

We want 0 in the second row, first column. Multiply row one by -3 and add to row two. We get

$\left[\begin{array}{cc|c} 1 & -4 & -6 \\ 0 & 13 & 13 \end{array}\right]$.

We now have the system $x - 4y = -6$
$13y = 13$.

Then $y = 1$ and replace y by 1 in the first equation $x - 4(1) = -6$
$x - 4 = -6$
$x = -2$

The solution set is $\{(-2,1)\}$.

10. $-x - y = 4$
$2x + 2y = -1$ augmented matrix is $\left[\begin{array}{cc|c} -1 & -1 & 4 \\ 2 & 2 & -1 \end{array}\right]$

Multiply row 1 by 2 and add to row 2.

$\left[\begin{array}{cc|c} -1 & -1 & 4 \\ 0 & 0 & 7 \end{array}\right]$

Row 2 reads
$0x + 0y = 7$
$0 = 7$ (False)

The system is inconsistent and the solution set is $\emptyset$.

14. $x + 3y - z = 5$
$3x - y + 2z = 5$
$x + y + 2z = 7$ augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 3 & -1 & 2 & 5 \\ 1 & 1 & 2 & 7 \end{array}\right]$

Multiply row one by -3 and add to row two.

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & -10 & 5 & -10 \\ 1 & 1 & 2 & 7 \end{array}\right]$

Subtract row one from row three.

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & -10 & 5 & -10 \\ 0 & -2 & 3 & 2 \end{array}\right]$

Multiply row two by $-\frac{1}{10}$.

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & -2 & 3 & 2 \end{array}\right]$

Multiply row two by 2 and add to row three.

$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 5 \\ 0 & 1 & -\frac{1}{2} & 1 \\ 0 & 0 & 2 & 4 \end{array}\right]$

$x + 3y - z = 5$
$y - \frac{1}{2}z = 1$
$2z = 4$
$z = 2$

Replace z by 2 in $y - \frac{1}{2}z = 1$
$y - 1 = 1$
$y = 2$.

Replace z by 2 and y by 2 in $x + 3y - z = 5$.
$x + 3(2) - 2 = 5$
$x + 6 - 2 = 5$
$x + 4 = 5$
$x = 1$

The solution set is $\{(1,2,2)\}$.

Review exercises

1. x-intercept, $(-4,0)$; y-intercept, $(0,8)$ **2.** x-intercept, $(-3,0)$; y-intercept, $\left(0,\frac{9}{2}\right)$ **3.** x-intercept, none; y-intercept, $(0,6)$

4. $x^2 + 8x + 16 = (x + 4)^2$ **5.** $y^2 - 5y + \frac{25}{4} = \left(y - \frac{5}{2}\right)^2$

6. $z^2 - \frac{1}{2}z + \frac{1}{16} = \left(z - \frac{1}{4}\right)^2$ **7.** $(y - 7)^2$ **8.** $(x + 5)^2$

Chapter 8 review

1. $\{(3,1)\}$ **2.** $\left\{\left(\frac{1}{2},\frac{1}{3}\right)\right\}$ **3.** $\{(x,y)|2x + 3y = 4\}$; dependent

4. $\emptyset$; inconsistent **5.** $\left\{\left(\frac{5}{4},-\frac{1}{4}\right)\right\}$ **6.** $\left\{\left(-\frac{12}{5},\frac{26}{5}\right)\right\}$

7. $F_1 = \frac{39}{5}$, $F_2 = \frac{3}{5}$ **8.** $\{(3,-4)\}$ **9.** $\{(-1,-10)\}$

10. $\left\{\left(-\frac{2}{7},\frac{1}{7}\right)\right\}$ **11.** $\left\{\left(-\frac{11}{4},\frac{27}{4}\right)\right\}$ **12.** $\left\{\left(-\frac{7}{3},-\frac{5}{6}\right)\right\}$

13. $\left(\frac{11}{5},-\frac{3}{5}\right)$ **14.** $\ell = 60$ ft; $w = 30$ ft **15.** 4 false alarms; 26 real alarms **16.** \$1,800 at 6½%; \$1,200 at 7%

17. 16⅔ g of 20% tin; 33⅓ g of 5% tin **18.** \$72 for topcoat; \$105 for suit **19.** 5.2 hr **20.** $\{(1,2,3)\}$ **21.** $\left\{\left(\frac{1}{10},\frac{1}{2},-\frac{3}{10}\right)\right\}$

22. $\{(3,-5,-1)\}$ **23.** $\{(3,2,4)\}$ **24.** $a = 3, b = -8, c = 2$

25. $F_1 = \frac{35}{2}$, $F_2 = -5$, $F_3 = 25$ **26.** 41°, 57°, 82°

27. 135 nickels; 120 dimes; 30 quarters **28.** 7 **29.** 24 **30.** 41

31. 108 **32.** 39 **33.** 88 **34.** $\{(1,-1)\}$ **35.** $\left\{\left(-\frac{5}{26},\frac{2}{13}\right)\right\}$

36. $\left\{\left(\frac{15}{4},9\right)\right\}$ **37.** $\left\{\left(\frac{8}{3},6\right)\right\}$ **38.** $\{(0,0,0)\}$

39. $\left\{\left(\frac{7}{6},\frac{-5}{6},\frac{-7}{6}\right)\right\}$ **40.** $\left\{\left(\frac{4}{7},-\frac{2}{7}\right)\right\}$ **41.** $\left\{\left(-3,\frac{16}{5},\frac{23}{5}\right)\right\}$

Chapter 8 cumulative test

1. -258 **2.** $V = 110$ **3.** $C = \frac{24}{5}$ **4.** $3a^2 - 2a + 6 + \frac{15}{a - 3}$

5. $6a^5 - 5a^3 + 3a$ **6.** $4x^2 - 56x - 45$ **7.** $\frac{2x^4}{y^5}$ **8.** $\left\{\frac{14}{13}\right\}$

9. $\left\{\frac{1 + \sqrt{13}}{3},\frac{1 - \sqrt{13}}{3}\right\}$ **10.** $\{-3 + \sqrt{14},-3 - \sqrt{14}\}$

11. $\left\{\frac{-1 + \sqrt{41}}{4},\frac{-1 - \sqrt{41}}{4}\right\}$ **12.** $\left\{z|z \geq -\frac{4}{3}\right\} = \left[-\frac{4}{3},\infty\right)$

13. $\{x|-3 \leq x \leq 1\} = [-3,1]$ **14.** $\left\{y|y < -5 \text{ or } y > \frac{1}{2}\right\}$ $= (-\infty,-5) \cup \left(\frac{1}{2},\infty\right)$ **15.** $\left\{\frac{5}{2},-\frac{3}{2}\right\}$

16. $\left\{x|x < \frac{1}{2} \text{ or } x > \frac{7}{6}\right\} = \left(-\infty,\frac{1}{2}\right) \cup \left(\frac{7}{6},\infty\right)$

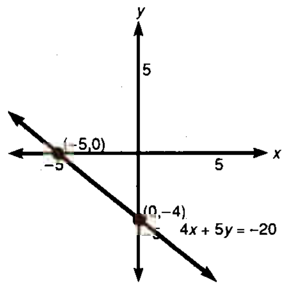
17. $\frac{4}{2y - 3}$ **18.** $\frac{4x^2 - 13x - 12}{3}$ **19.** $\frac{4x + 2}{(x - 5)(x + 4)(x - 3)}$

20. $\frac{2y}{y - 7}$ **21.** $\frac{3abp^3}{14q}$ **22.** 9 **23.** $\frac{1}{y^3}$ **24.** $a^{5/4}$ **25.** $4\sqrt{5}$

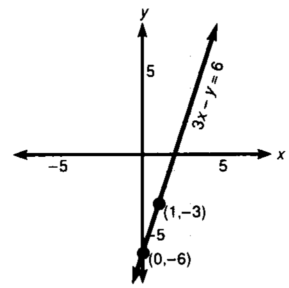
26. $2\sqrt{3} + 3\sqrt{2} - 4\sqrt{6} - 12$ **27.** $-9 + 40i$ **28.** $-2y\sqrt[3]{x}$

29. $\emptyset$; 36 and 1 are extraneous

30.

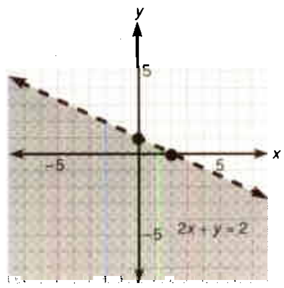


31.



32. $5x + 7y = 28$

33. $x + 4y = 13$ **34.**



35. -31 **36.** $\{(2,0)\}$ **37.** $\left\{\left(\frac{9}{5},-\frac{11}{5}\right)\right\}$

38. $\left\{\left(\frac{3}{17},-\frac{4}{17}\right)\right\}$ **39.** $\left\{\left(-\frac{5}{11},-\frac{7}{11}\right)\right\}$

40. $\left\{\left(-\frac{1}{6},-\frac{2}{3},\frac{1}{6}\right)\right\}$

Chapter 9

Exercise 9–1

Answers to odd-numbered problems

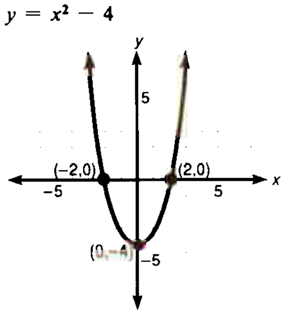
1. (3,4) **3.** $(0,-16)$ **5.** (5,0) **7.** $(-2,-9)$ **9.** (1,4)

11. $\left(\frac{7}{4}, -\frac{25}{8}\right)$ **13.** x-intercepts, none; y-intercept, (0,13)

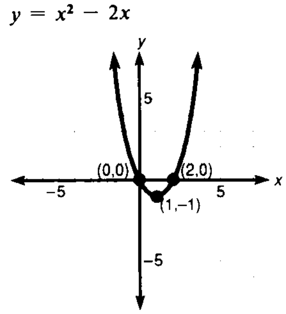
15. x-intercepts, $(-4,0)$, (4,0); y-intercept, $(0,-16)$

17. x-intercept, (5,0); y-intercept, (0,25) **19.** x-intercepts, $(-5,0)$, (1,0); y-intercept, $(0,-5)$ **21.** x-intercepts, (3,0), $(-1,0)$; y-intercept, (0,3) **23.** x-intercepts, (3,0), $\left(\frac{1}{2}, 0\right)$; y-intercept, (0,3)

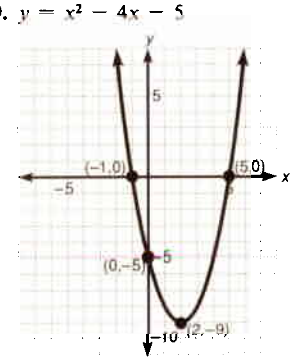
25. $y = x^2 - 4$



27. $y = x^2 - 2x$



29. $y = x^2 - 4x - 5$



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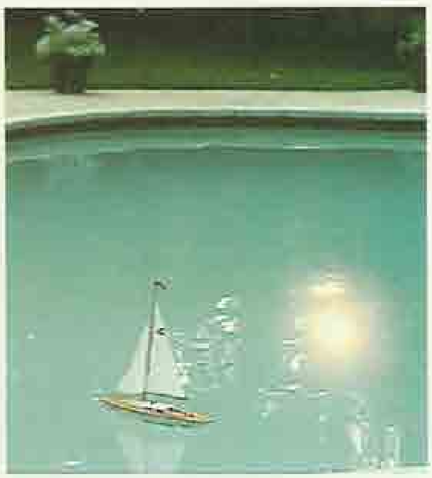

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